

NON AUTONOMOUS PARABOLIC PROBLEMS WITH UNBOUNDED COEFFICIENTS IN UNBOUNDED DOMAINS

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ABSTRACT. Given a class of nonautonomous elliptic operators $\mathcal{A}(t)$ with unbounded coefficients, defined in $\overline{I \times \Omega}$ (where I is a right-halfline or $I = \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$ is possibly unbounded), we prove existence and uniqueness of the evolution operator associated to $\mathcal{A}(t)$ in the space of bounded and continuous functions, under Dirichlet and first order, non tangential homogeneous boundary conditions. Some qualitative properties of the solutions, the compactness of the evolution operator and some uniform gradient estimates are then proved.

1. INTRODUCTION

Parabolic Cauchy problems with unbounded coefficients set in unbounded domains, with sufficiently smooth boundary, have been studied in the autonomous case both in the case of homogeneous Dirichlet [8] and Neumann [5, 6] boundary conditions. On the other hand, the nonautonomous counterpart have been studied, to the best of our knowledge, only in the particular case $\Omega = \mathbb{R}_+^d$, again only under homogeneous Dirichlet and Neumann boundary conditions [3].

This paper is devoted to continue the analysis started in [3], studying parabolic nonautonomous boundary Cauchy problems with unbounded coefficients in a greater generality, with respect to both the domain, where the Cauchy problems are set, and the boundary conditions considered. More precisely, let $\Omega \subset \mathbb{R}^d$ be an unbounded open set with a boundary of class $C^{2+\alpha}$, for some $\alpha \in (0, 1)$, and let $I \subset \mathbb{R}$ be an open right halfline (possibly $I = \mathbb{R}$). For any fixed $s \in I$ and any $f \in C_b(\Omega)$ (the space of bounded and continuous functions on Ω), we consider the nonautonomous Cauchy problem

$$\begin{cases} D_t u(t, x) = (\mathcal{A}u)(t, x), & t \in (s, +\infty), x \in \Omega, \\ (\mathcal{B}u)(t, x) = 0, & t \in (s, +\infty), x \in \partial\Omega, \\ u(s, x) = f(x), & x \in \Omega. \end{cases} \quad (P_{\mathcal{B}})$$

The families of nondegenerate elliptic operators $\{\mathcal{A}(t)\}_{t \in I}$ and of boundary operators $\{\mathcal{B}(t)\}_{t \in I}$ act on smooth functions ζ as follows:

$$(\mathcal{A}(t)\zeta)(x) = \sum_{i,j=1}^d q_{ij}(t, x) D_{ij}\zeta(x) + \sum_{i=1}^d b_i(t, x) D_i\zeta(x) - c(t, x)\zeta(x), \quad (1.1)$$

for any $(t, x) \in I \times \Omega$, and

$$(\mathcal{B}(t)\zeta)(x) = \sum_{i=1}^d \beta_i(t, x) D_i\zeta(x) + \gamma(t, x)\zeta(x), \quad (t, x) \in I \times \partial\Omega. \quad (1.2)$$

2000 *Mathematics Subject Classification.* 35K10, 35K15, 35B65.

Key words and phrases. nonautonomous second-order elliptic operators, unbounded coefficients, evolution operators, compactness, invariant subspaces.

The authors are members of GNAMPA of the Italian Istituto Nazionale di Alta Matematica. This work has been supported by the M.I.U.R. Research Project PRIN 2010-2011 “Problemi differenziali di evoluzione: approcci deterministici e stocastici e loro interazioni” and INdAM-GNAMPA Project 2014 “Equazioni ellittiche e paraboliche a coefficienti illimitati”.

The coefficients of the previous operators are smooth enough functions, and all of them but $\beta = (\beta_1, \dots, \beta_d)$ may be unbounded; function β either everywhere differs from 0 on $\partial\Omega$ or therein identically vanishes. In the first case, we assume the usual non-tangential condition, in the latter one, we assume that $\gamma \equiv 1$ so that $\mathcal{B}\zeta$ is the trace of ζ on $\partial\Omega$.

We first prove existence and uniqueness of a bounded classical solution of problem $(P_{\mathcal{B}})$ (see Definition 3.1). The case $\gamma \geq 0$ requires rather weak assumptions on the coefficients of the operators $\mathcal{A}(t)$ and $\mathcal{B}(t)$. No growth assumptions are assumed on the diffusion and drift coefficients of the operators $\mathcal{A}(t)$, whereas the potential is assumed to be bounded from below, this condition being not surprising at all since, as the autonomous case reveals: without any lower bound on the potential no bounded solutions to problem $(P_{\mathcal{B}})$ exist in general. Further, the existence of a so-called Lyapunov function φ , associated with the pair $(\mathcal{A}(t), \mathcal{B}(t))$ (cf. Hypothesis 2.4) is assumed, which serves as a fundamental tool to prove a maximum principle, which yields uniqueness of the solution to problem $(P_{\mathcal{B}})$. When γ takes also negative values we assume an extra condition, which is stated in terms of another Lyapunov function. The existence and the uniqueness of a classical solution to problem $(P_{\mathcal{B}})$ allow us to define an evolution operator $G_{\mathcal{B}}(t, s)$ of bounded linear operators in $C_b(\Omega)$ and to prove some remarkable continuity properties that this evolution operator enjoys. As a consequence of the Riesz representation theorem and the continuity property of the evolution operator, we can show that, for any $(t, s) \in \Lambda := \{(t, s) \in I \times I : t > s\}$ and any $x \in \Omega$, there exists a finite Borel measure $g_{\mathcal{B}}(t, s, x, dy)$ such that

$$(G_{\mathcal{B}}(t, s)f)(x) = \int_{\Omega} f(y)g_{\mathcal{B}}(t, s, x, dy), \quad f \in C_b(\Omega). \quad (1.3)$$

Under an additional smoothness assumption on the diffusion coefficients we prove that $G_{\mathcal{B}}(t, s)f$ admits an integral representation by means of a Green function $g_{\mathcal{B}} : \Lambda \times \Omega \times \Omega \rightarrow (0, +\infty)$, i.e., $g_{\mathcal{B}}(t, s, x, dy) = g_{\mathcal{B}}(t, s, x, y)dy$ for any $(t, s, x, y) \in \Lambda \times \Omega \times \Omega$. For any fixed $s \in I$ and almost any $y \in \Omega$, the function $g_{\mathcal{B}}(\cdot, s, \cdot, y)$ is smooth, satisfies $D_t g_{\mathcal{B}} - \mathcal{A}(t)g_{\mathcal{B}} = 0$ in $(s, +\infty) \times \Omega$.

Formula (1.3) plays a crucial role in the study of the compactness of the operator $G_{\mathcal{B}}(t, s)$ in $C_b(\Omega)$. Indeed, as the proof of Theorem 4.5 reveals, the compactness of the operators $G_{\mathcal{B}}(t, s)$ in $C_b(\Omega)$, for $(t, s) \in \Lambda \times J^2$, J being a bounded interval, follows from the tightness of the family of measures $\{g_{\mathcal{B}}(t, s, x, dy), x \in \Omega\}$ for any $(t, s) \in \Lambda \cap J^2$. In view of this fact, a sufficient condition is then provided to guarantee the tightness of the previous family of measures. Our result extends the results obtained in [2, 13] in the case when $\Omega = \mathbb{R}^d$.

Next, when the boundary operator \mathcal{B} is independent of t , under some growth assumptions on the coefficients q_{ij} , b_i and c at infinity and assuming that they are bounded in a small neighborhood of $\partial\Omega$, we prove a uniform gradient estimate for $G_{\mathcal{B}}(t, s)f$. More precisely, we show that for any $T > s \in I$, there exists a positive constant $C_{s,T}$ such that

$$\|\nabla_x G_{\mathcal{B}}(t, s)f\|_{\infty} \leq \frac{C_{s,T}}{\sqrt{t-s}} \|f\|_{\infty}, \quad t \in (s, T), \quad (1.4)$$

for any $f \in C_b(\Omega)$. Estimate (1.4) (which can be then extended, by the evolution law, to all $t \in (s, +\infty)$) is classical when the coefficients of $\mathcal{A}(t)$ are bounded and Ω is an open set with sufficiently smooth boundary, either bounded or unbounded (see [12]). Recently, it has been proved for the semigroup $T(t)$ associated in $C_b(\Omega)$ to autonomous elliptic operators with unbounded coefficients, both in the case of homogeneous Neumann (first in convex sets [5] and, then, in the general case [6]) and Dirichlet boundary conditions [8]. Very recently, we proved estimate (1.4) for

the solution to problem (P_B) in \mathbb{R}_+^d when homogeneous Dirichlet and Neumann boundary conditions are prescribed on $\partial\mathbb{R}_+^d$. The simple geometry of \mathbb{R}_+^d and suitable assumptions on the coefficients of the operator $\mathcal{A}(t)$, allowed to extend these latter ones to \mathbb{R}^d and to reduce the problem to the whole space \mathbb{R}^d , where gradient estimates were already known ([10]). A symmetry argument was then used to come back to the Neumann and Dirichlet Cauchy problems set in \mathbb{R}_+^d .

In our situation the key tools to prove (1.4) are the Bernstein method, the maximum principle in Proposition 3.2 and the geometric Lemma A.2 which allows to locally transform the boundary Cauchy problem (P_B) into a Cauchy problem in the halfspace \mathbb{R}_+^d where homogeneous Robin boundary conditions are prescribed. Bernstein method works very well in the whole space and it is easy to explain: one considers the function $t \mapsto v(t, \cdot) = (G(t, s)f)^2 + a(t - s)|\nabla_x G(t, s)f|^2$ and shows that, under suitable assumptions and a suitable choice of the positive parameter a , $D_t v - \mathcal{A}(t)v \leq 0$. A variant of the maximum principle reveals that the supremum of function v is attained on $\{s\} \times \mathbb{R}^d$, and the gradient estimate follows at once. When \mathbb{R}^d is replaced by an open set Ω , things become much more difficult. Indeed, the supremum of v could be attained on $\partial\Omega$. Hence, one needs to bound the suprema of v on $\partial\Omega$. In the autonomous case, this has been done in the case of Dirichlet and Neumann boundary conditions. In the first case an a priori gradient estimate on the boundary of Ω has been proved by a comparison argument, which reveals that therein the function $t \mapsto \sqrt{t}|\nabla_x T(t)f|$ can be bounded uniformly by a constant times the sup-norm of f . The argument in [8] can not be adapted to the case of different boundary conditions. Neumann boundary conditions have been considered first in convex domains (see [5]), where the geometry of Ω shows that the normal derivative of $|\nabla_x T(t)f|^2$ is nonpositive, so that the normal derivative of v is nonpositive on $\partial\Omega$ as well and, consequently, the supremum of v is attained on $\{0\} \times \Omega$. When Ω is nonconvex, the normal derivative of $|\nabla_x T(t)f|^2$ does not need to be nonnegative. But, as in [6], replacing v by the function $t \mapsto w(t, \cdot) = (T(t)f)^2 + am|\nabla_x T(t)f|^2$ for a suitable function m , which takes into account the curvatures of $\partial\Omega$, one can still prove that $D_t w - \mathcal{A}w$ and the normal derivative of w , are nonnegative in Ω and $\partial\Omega$, respectively.

Clearly, for more general unbounded domains and more general boundary conditions, the same arguments do not work, therefore we need to develop new strategies to prove the uniform gradient estimate (1.4). Here, the idea is to use the regularity of the domain to go back by means of local charts to problems defined in \mathbb{R}_+^d or in \mathbb{R}^d . Assuming more smoothness on the domain Ω and the vector β , we determine coordinate transformations which, locally transform the homogeneous boundary condition $\mathcal{B}u = 0$ on the boundary $\partial\Omega$ to an homogeneous Robin boundary condition on $\mathbb{R}^{d-1} \times \{0\}$. Thus, under the assumption that the coefficients of $\mathcal{A}(t)$ are bounded only in a neighborhood of the boundary $\partial\Omega$, we prove an uniform gradient estimates in a small strip Ω_δ near the boundary. Finally, some growth assumptions on the diffusion coefficients and the potential term and a quite standard dissipativity condition on the drift term b , are enough to show that (1.4) is satisfied also in $\Omega \setminus \Omega_\delta$. We point out that, differently from [5, 6, 8], we do not assume that the diffusion coefficients q_{ij} are globally bounded together with their spatial gradients. Moreover, our results seem to be new also in the autonomous case when \mathcal{B} is a general first-order boundary operator. In particular, we can cover also the case when γ changes sign on $\partial\Omega$.

The special case when Ω is convex and homogeneous Neumann boundary conditions are prescribed, can be treated and estimate (1.4) can be proved without assuming any additional smoothness assumption on the domain and any hypotheses of boundedness for the coefficients of $\mathcal{A}(t)$ in a neighborhood of the boundary.

This can be done adapting the arguments used in the autonomous case, described here above.

Also when $\Omega = \mathbb{R}_+^d$ and homogeneous Robin boundary conditions are prescribed on $\mathbb{R}^{d-1} \times \{0\}$, we do not need to assume that the drift term b and the potential term c are bounded. Indeed, a simple trick allows us to transform homogeneous Robin boundary condition into homogeneous Neumann condition on $\partial\mathbb{R}_+^d$. Hence, we are reduced to a problem set in a convex set with Robin boundary conditions, to which we can apply the already established results.

The paper is split into sections as follows. In Section 2 we state the main assumptions on the coefficients of the operators $\mathcal{A}(t)$ and $\mathcal{B}(t)$ and on the domain Ω , recalling also some consequences of the smoothness of the domain. In Section 3, we first prove a maximum principle for solutions to the problem $(P_{\mathcal{B}})$, which are continuous in $([s, +\infty) \times \overline{\Omega}) \setminus (\{s\} \times \partial\Omega)$. Then, we construct the solution to the problem $(P_{\mathcal{B}})$. In Section 4 we introduce the evolution operator $G_{\mathcal{B}}(t, s)$ and we investigate on some of its qualitative properties, such as compactness. Section 5 is devoted to prove the uniform gradient estimates (1.4) and in Section 6 we provide some examples of operators to which our results can be applied. The appendix collects some technical results used in the paper.

Notations. For any open set (or the closure of an open set) \mathcal{O} , any interval $J \subset \mathbb{R}$ and any $\delta > 0$, we set $\mathcal{O}_\delta := \{x \in \overline{\mathcal{O}} : r_{\mathcal{O}}(x) < \delta\}$ (where $r_{\mathcal{O}}(x) = \text{dist}(x, \partial\mathcal{O})$) and $\mathcal{O}_J := J \times \mathcal{O}$. Further, by $\nu(x)$ we mean the outward unit normal to $\partial\mathcal{O}$ at x .

We assume that the reader is familiar with the spaces $C^k(\mathcal{O})$ ($k \geq 0$) and $C^{\alpha, \beta}(\mathcal{O}_J)$ ($\alpha, \beta \geq 0$). By $C_b^k(\mathcal{O})$ we denote the subspace of $C^k(\mathcal{O})$ consisting of functions which are bounded together with all existing derivatives. We use the subscript “ c ” (resp. “0”) for spaces of functions with compact support (resp. for spaces of functions vanishing on $\partial\mathcal{O}$ and at infinity). When $k \in (0, 1)$, we write $C_{\text{loc}}^k(\mathcal{O})$ to denote the space of all $f \in C(\mathcal{O})$ which are Hölder continuous in any compact set of \mathcal{O} . Analogously, we define the spaces $C_{\text{loc}}^{\alpha/2, \alpha}(\mathcal{O}_J)$ and $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\mathcal{O}_J)$ ($\alpha \in (0, 1)$).

The notations $D_t f := \frac{\partial f}{\partial t}$, $D_i f := \frac{\partial f}{\partial x_i}$, $D_{ij} f := \frac{\partial^2 f}{\partial x_i \partial x_j}$ are extensively used, as well as the notation $J_x f$, to denote the Jacobian matrix, with respect to the spatial variables, of the function $f : \mathcal{O}_J \rightarrow \mathbb{R}^d$. χ_A denotes the characteristic function of the set $A \subset \mathcal{O}$ and $\mathbb{1} := \chi_{\mathcal{O}}$. The Euclidean ball with center at x_0 and radius $R > 0$ is denoted by $B_R(x_0)$, $B_R := B_R(0)$ and $B_R^+ := B_R \cap \mathbb{R}_+^d$. Similarly, \mathcal{O}^R denotes the set $\mathcal{O} \cap B_R$. Occasionally, we find it convenient to split $\mathbb{R}^d \ni x = (x', x_d)$ with $x_d \in \mathbb{R}$. Finally, $a^+ := \max\{a, 0\}$ for any $a \in \mathbb{R}$.

2. MAIN ASSUMPTIONS AND PRELIMINARIES

Let $I \subset \mathbb{R}$ be an open right halfline (possibly $I = \mathbb{R}$) and Ω be a domain of \mathbb{R}^d . Let us introduce our standing assumptions on the domain Ω and on the coefficients of the operators $\mathcal{A}(t)$ in (1.1):

- Hypotheses 2.1.** (i) $\partial\Omega$ is uniformly of class $C^{2+\alpha}$ for some $0 < \alpha < 1$;
(ii) q_{ij} , b_i and c belong to $C_{\text{loc}}^{\alpha/2, \alpha}(\overline{\Omega_I})$ for every $i, j = 1, \dots, d$;
(iii) $c_0 := \inf_{\Omega_I} c \geq 0$;
(iv) Q is uniformly elliptic, i.e., for every $(t, x) \in \Omega_I$, the matrix $Q(t, x)$ is symmetric and there exists a function $\eta : \Omega_I \rightarrow \mathbb{R}^+$ such that $0 < \eta_0 := \inf_{\Omega_I} \eta$ and $\langle Q(t, x)\xi, \xi \rangle \geq \eta(t, x)|\xi|^2$ for any $\xi \in \mathbb{R}^d$ and $(t, x) \in \Omega_I$.

Remark 2.2. (a) Hypothesis 2.1(i) is standard when problems are defined on unbounded domains. It means that

- (i) there exist $R > 0$, a (at most countable) collection of open balls $B_R(x_h) =: V_h$, $h \in \mathbb{N}$, covering $\partial\Omega$, and $k \in \mathbb{N}$ such that $\sum_{h=1}^{+\infty} \chi_{V_h} \leq k$ in \mathbb{R}^d , i.e., $\bigcap_{h \in H} V_h = \emptyset$ if $H \subset \mathbb{N}$ contains more than k elements;
 - (ii) there exist coordinate transformations $\psi_h : V_h \rightarrow B_1$ ($h \in \mathbb{N}$), which are $C^{2+\alpha}$ -diffeomorphisms such that $\psi_h(V_h \cap \Omega) = B_1^+$, $\psi_h(V_h \cap \partial\Omega) = B_1 \cap \partial\mathbb{R}_+^d$ for each h , and $\sup_{h \in \mathbb{N}} (\|\psi_h\|_{C^{2+\alpha}(V_h)} + \|\psi_h^{-1}\|_{C^{2+\alpha}(B_1)}) < +\infty$;
 - (iii) there exists $\varepsilon > 0$ such that $\bigcup_{h \in \mathbb{N}} B_{R/2}(x_h) \supset \Omega_\varepsilon$.
- (b) The smoothness of $\partial\Omega$ implies that the distance function r_Ω belongs to $C_b^2(\Omega_\delta)$ for some $\delta > 0$. For any $x \in \Omega_\delta$, it holds that $\nabla r_\Omega(x) = -\nu(\pi(x))$, where $\pi(x)$ is the projection of x on $\partial\Omega$. Finally, the equiboundedness of the $C^{2+\alpha}$ -norms of ψ_h and ψ_h^{-1} , shows that $\kappa = \inf_{x \in \partial\Omega} \{ \langle J\nu(x)\tau, \tau \rangle : |\tau| = 1, \langle \tau, \nu(x) \rangle = 0 \} \in \mathbb{R}$.
- (c) Since the last component ψ_h^d of the function ψ_h ($h \in \mathbb{N}$) identically vanishes on $\partial\Omega \cap B_R(x_h)$ and it is positive inside $\Omega \cap V_h$, $\nabla \psi_h^d = -|\nabla_x \psi_h^d| \nu$ in $\partial\Omega \cap B_R(x_h)$.

As far as the boundary operators $\mathcal{B}(t)$ in (1.2) are concerned, when $\beta \equiv 0$, we assume that $\gamma \equiv 1$ in order to recover the Cauchy Dirichlet problem. On the other hand when $\beta \not\equiv 0$, we assume the following assumptions on the coefficients of $\mathcal{B}(t)$.

Hypotheses 2.3. (i) β_i ($i = 1, \dots, d$) and γ belong to $C_{\text{loc}}^{(1+\alpha)/2, 1+\alpha}(\bar{I} \times \partial\Omega)$;
(ii) γ is bounded from below and $|\beta| \equiv 1$ in $I \times \partial\Omega$;
(iii) $\inf_{(t,x) \in [a,b] \times \partial\Omega} \langle \beta(t,x), \nu(x) \rangle > 0$ for any $[a,b] \subset I$.

To guarantee the uniqueness of the bounded classical solution to the problem (P_B) (see Definition 3.1), we assume the following condition.

Hypotheses 2.4. (i) For any bounded interval $J \subset I$ there exist a positive function $\varphi = \varphi_J \in C^2(\overline{\Omega_J})$ and a positive number $\lambda = \lambda_J$ such that φ blows up as $|x| \rightarrow +\infty$, uniformly with respect to $t \in J$, and $D_t\varphi - \mathcal{A}\varphi + \lambda\varphi > 0$ in Ω_J .
(ii) When $\beta \not\equiv 0$, we require in addition that $\mathcal{B}\varphi \geq 0$ in $J \times \partial\Omega$.

Remark 2.5. Actually, the condition on the sign of c_0 is not restrictive; Hypotheses 2.1(iii) can be replaced by the assumption that $c_0 > -\infty$. Indeed, if $c_0 < 0$, and u solves problem (P_B) then the function $(t,x) \mapsto \tilde{u}(t,x) = e^{c_0(t-s)}u(t,x)$, which has the same regularity as u , satisfies $D_t\tilde{u} - \mathcal{A}_0\tilde{u} = 0$, where $\mathcal{A}_0u = \mathcal{A}u + c_0u$ has a nonnegative zero-order coefficient. Moreover, \mathcal{A}_0 satisfies Hypotheses 2.4 with the same Lyapunov function φ and the same positive constant λ .

3. EXISTENCE AND UNIQUENESS

Here, we prove existence and uniqueness of the bounded classical solution to problem (P_B) . Throughout this section, we denote by S the set $\{s\} \times \partial\Omega$.

Definition 3.1. A function u is called a bounded classical solution of the problem (P_B) if $u \in C^{1,2}(\Omega_{(s,+\infty)}) \cap C_b(\overline{\Omega_{[s,+\infty)}} \setminus S)$ and satisfies (P_B) .

3.1. The case when $\gamma \geq 0$. The uniqueness of the classical solution to problem (P_B) is a consequence of suitable maximum principle.

Proposition 3.2. Let $T > s \in I$ and $u \in C^{1,2}(\Omega_{(s,T)}) \cap C_b(\overline{\Omega_{(s,T)}} \setminus S)$ satisfy

$$\begin{cases} D_t u(t,x) - (\mathcal{A}u)(t,x) \leq 0, & (t,x) \in \Omega_{(s,T)}, \\ (\mathcal{B}u)(t,x) \leq 0, & (t,x) \in (s,T) \times \partial\Omega, \\ u(s,x) \leq 0, & x \in \Omega. \end{cases} \quad (3.1)$$

Then, $u \leq 0$ in $\Omega_{(s,T)}$.

Proof. Let $\lambda = \lambda_{[s,T]}$ and $\varphi = \varphi_{[s,T]}$ be the constant and the function in Hypothesis 2.4. Up to replacing λ with a larger value, if needed, we can assume that $D_t\varphi - \mathcal{A}\varphi + \lambda\varphi > 0$ in $\Omega_{(s,T)}$. To prove that $u \leq 0$ in $\Omega_{(s,T)}$, for any $n \in \mathbb{N}$ we introduce the function v_n , defined by $v_n(t, x) = e^{-\lambda(t-s)}u(t, x) - n^{-1}\varphi(t, x)$ for any $(t, x) \in \overline{\Omega_{(s,T)}} \setminus S$, and prove that v_n is nonpositive. Then, letting $n \rightarrow +\infty$ we conclude that u is nonpositive as well.

Since φ tends to $+\infty$ as $|x| \rightarrow +\infty$, uniformly with respect to $t \in [s, T]$, and u is bounded, v_n tends to $-\infty$ as $|x| \rightarrow +\infty$, uniformly with respect to $t \in [s, T]$, for any $n \in \mathbb{N}$. We can thus fix $R > 0$ large enough such that $v_n < 0$ in $[s, T] \times (\Omega \setminus B_R)$. It thus follows that we just need to prove that $v_n \leq 0$ in $(s, T] \times \Omega^R$.

We split the rest of the proof in two steps. In the first one, we assume that u is continuous in the whole of $\Omega_{(s,T)}$. Then, in Step 2, we consider the general case.

Step 1. Since u is continuous in $\overline{\Omega_{(s,T)}}$, v_n satisfies

$$\begin{cases} D_tv_n(t, x) - (\mathcal{A}v_n)(t, x) + \lambda v_n(t, x) < 0, & (t, x) \in (s, T] \times \Omega^R, \\ (\mathcal{B}v_n)(t, x) \leq 0, & (t, x) \in (s, T] \times \partial_1\Omega^R, \\ v_n(t, x) < 0, & (t, x) \in (s, T] \times \partial_2\Omega^R, \\ v_n(s, x) < 0, & x \in \overline{\Omega^R}, \end{cases} \quad (3.2)$$

where $\partial\Omega^R = \partial_1\Omega^R \cup \partial_2\Omega^R := (\partial\Omega \cap B_R) \cup (\overline{\Omega} \cap \partial B_R)$. We follow the lines in the proof of [9, Thm. 2.16]. For this purpose, we introduce the set $\mathcal{J}_n := \{r \in [s, T] : v_n < 0 \text{ in } \overline{\Omega_{(s,r)}^R}\}$, which contains s . Since u and φ are continuous in $\overline{\Omega_{(s,T)}}$, the function v_n is uniformly continuous in $\overline{\Omega_{(s,T)}^R}$. This implies that \mathcal{J}_n is an interval and $\sup \mathcal{J}_n > s$. Let us denote by τ_n the supremum of \mathcal{J}_n and prove that $\tau_n = T$. By contradiction, we assume that $\tau_n < T$. Then, by continuity $v_n(\tau_n, \cdot) \leq 0$ in $\overline{\Omega^R}$ and there exists $x_n \in \overline{\Omega^R}$ such that $v_n(\tau_n, x_n) = 0$. The point (τ_n, x_n) turns out to be the maximum point of the restriction of v_n to $\overline{\Omega_{(s,\tau_n)}^R}$. Moreover, x_n can not belong to Ω^R , otherwise we would have $(\mathcal{A}v_n)(\tau_n, x_n) - \lambda v_n(\tau_n, x_n) \leq 0$ and $D_tv_n(\tau_n, x_n) \geq 0$, thus contradicting (3.2). Hence, $x_n \in \partial\Omega^R$. Actually, x_n can not belong to $\partial_2\Omega^R$ and, clearly, it can not belong to $\partial_1\Omega^R$, if $\mathcal{B} \equiv I$. Indeed, in this case $\mathcal{B}v_n = -n^{-1}\varphi$, which is negative (see Hypothesis 2.4(i)). On the other hand, if \mathcal{B} is a first-order boundary operator and $x_n \in \partial_1\Omega^R \subset \partial\Omega$, then we would have $\langle \beta(\tau_n, x_n), \nabla_x v_n(\tau_n, x_n) \rangle > 0$, since at each point of $\partial\Omega$ the interior sphere condition is satisfied (see e.g., [14, Thm. 3.7]). But this contradicts the boundary condition in (3.2). We thus conclude that $\tau = T$ so that v_n is negative in $\overline{\Omega_{(s,T)}^R}$.

Step 2. We now consider the general case when u is not continuous on S . Since the above arguments do not work, we use a different strategy and we adapt to our situation an idea which has been already used in [8, Thm. A.2] in the case of autonomous Dirichlet Cauchy problems. For any $n \in \mathbb{N}$, we introduce the function $w_{n,\varsigma} = v_n - M_n \varsigma^{\varepsilon\eta_0} \psi_\varsigma$, where $M_n = \sup_{\Omega_{(s,T)}^R} v_n$,

$$\psi_\varsigma(t, x) = \frac{1}{(t + \varsigma - s)^{\varepsilon\eta_0}} \exp\left(t + \varsigma - s - \frac{\varepsilon\omega^2(x)}{t + \varsigma - s}\right), \quad (t, x) \in \Omega_{(s,T)},$$

$\omega = \vartheta r_\Omega + 1 - \vartheta$, where $\vartheta \in C_b^2(\overline{\Omega})$ satisfies $\chi_{\Omega_{\delta/2}} \leq \vartheta \leq \chi_{\Omega_\delta}$ and δ is sufficiently small to have $r_{\mathbb{R}^d \setminus \Omega} \in C_b^2(\overline{\Omega_\delta})$ (see Remark 2.2). Finally, ς and ε are positive parameters. Function $w_{n,\varsigma}$ has the same regularity as u . We claim that

- (i) $D_tw_{n,\varsigma} - \mathcal{A}w_{n,\varsigma} + \lambda w_{n,\varsigma} \leq 0$ in $\Omega_{(s,T)}^R$ and $\mathcal{B}w_{n,\varsigma} \leq 0$ in $(s, T] \times \partial\Omega$, for suitably fixed $\varepsilon > 0$ and any $\varsigma \in (0, 1)$;
- (ii) there exists $\tau = \tau(\varsigma)$ such that $w_{n,\varsigma} \leq 0$ in $\Omega_{(s,s+\tau)}^R$.

Since $w_{n,\varsigma}$ is continuous in $\overline{\Omega_{(s+\tau,T)}}$, we can then apply Step 1 to show that $w_{n,\varsigma} \leq 0$ in $\overline{\Omega_{(s+\tau,T)}^R}$ and we conclude that $w_{n,\varsigma} \leq 0$ in $\Omega_{(s,T)}^R$. Letting $\varsigma \rightarrow 0^+$ we deduce that $v_n \leq 0$ in $\Omega_{(s,T)}^R$.

To check property (i), we prove that there exists $\varepsilon > 0$ such that, for any $\varsigma \in (0, 1)$, $D_t\psi_\varsigma - \mathcal{A}\psi_\varsigma + \lambda\psi_\varsigma$ and $\mathcal{B}\psi_\varsigma$ are positive in $\Omega_{(s,T)}^R$ and in $(s, T] \times \partial\Omega$, respectively. Observe that $D_t\psi_\varsigma - \mathcal{A}\psi_\varsigma + \lambda\psi_\varsigma$ is positive in $\Omega_{(s,T)}^R$ if and only if the function h_ς , defined by

$$h_\varsigma(t, \cdot) = (\lambda + 1 + c(t, \cdot))(t + \varsigma - s)^2 - \varepsilon\eta_0(t + \varsigma - s) - 4\varepsilon^2\omega^2\langle Q(t, \cdot)\nabla\omega, \nabla\omega \rangle \\ + \varepsilon\omega^2 + 2\varepsilon(t + \varsigma - s)\langle Q(t, \cdot)\nabla\omega, \nabla\omega \rangle + 2\varepsilon(t + \varsigma - s)\omega(\mathcal{A} + c)\omega,$$

for any $t \in (s, T)$, is positive in $\Omega_{(s,T)}^R$. To estimate the sign of the function h_ς , we denote by K_0 and K_1 the supremum over $\Omega_{(s,T)}^R$ of the functions $\langle Q\nabla\omega, \nabla\omega \rangle$ and $\langle b, \nabla\omega \rangle + \text{Tr}(QD^2\omega)$, respectively, and observe that $\Omega = \{x \in \Omega : |\nabla\omega(x)| > 3/4\} \cup \{x \in \Omega : \omega(x) \geq \sigma\} =: A \cup B$, for a suitable $\sigma > 0$. Indeed, for any $x \in \Omega$, $|\nabla\omega(x)| \geq |\nabla\omega(\xi)| - |\nabla\omega(x) - \nabla\omega(\xi)| \geq 1 - \|\omega\|_{C_b^2(\Omega_\delta)}|x - \xi|$, where ξ is the unique projection of x on $\partial\Omega$. Hence, $|\nabla\omega| \geq \frac{3}{4}$ in Ω_σ if $\sigma = (4\|\omega\|_{C_b^2(\Omega_\delta)})^{-1}$. Now, using the inequalities

$$2\varepsilon(t + \varsigma - s)\omega(\langle b(t, \cdot), \nabla\omega \rangle + \text{Tr}(Q(t, \cdot)D^2\omega)) \geq -\varepsilon^{\frac{3}{2}}\omega^2 - (t + \varsigma - s)^2K_1^2\sqrt{\varepsilon},$$

$$\langle Q\nabla\omega, \nabla\omega \rangle \geq \frac{9}{16}\eta_0\chi_A,$$

$$-\varepsilon(t + \varsigma - s) = -\varepsilon(t + \varsigma - s)\chi_A - \varepsilon(t + \varsigma - s)\chi_B \\ \geq -\varepsilon(t + \varsigma - s)\chi_A - \frac{1}{2}\varepsilon^{\frac{3}{2}}\chi_B - \frac{1}{2}(t + \varsigma - s)^2\sqrt{\varepsilon}\chi_B,$$

and, recalling that $c \geq 0$, we can estimate

$$h_\varsigma \geq \left(\lambda + 1 - \frac{1}{2}\sqrt{\varepsilon}\chi_B - K_1^2\sqrt{\varepsilon}\right)(\cdot + \varsigma - s)^2 + \varepsilon\omega^2[1 - \sqrt{\varepsilon}(1 + 4K_0\sqrt{\varepsilon})] - \frac{1}{2}\varepsilon^{\frac{3}{2}}\chi_B.$$

It is now clear that, if $\varepsilon \leq \varepsilon_0 := \min\{(\lambda + 1)^2K_1^{-4}, (8K_0)^{-2}(-1 + \sqrt{1 + 16K_0})^2\}$, then $h_\varsigma(t, \cdot)$ is nonnegative in A . On the other hand, $\omega \geq \sigma$ in B . Therefore, if $\varepsilon \leq (8K_0)^{-2}(-1 + \sqrt{1 + 8K_0})^2$, it holds that

$$\varepsilon\omega^2[1 - \sqrt{\varepsilon}(1 + 4K_0\sqrt{\varepsilon})] - \frac{1}{2}\varepsilon^{\frac{3}{2}} \leq \frac{1}{2}\varepsilon(\sigma^2 - \sqrt{\varepsilon}).$$

It is now clear that, if $\varepsilon \leq \varepsilon_1 := \min\{(8K_0)^{-2}(-1 + \sqrt{1 + 8K_0})^2, 4(\lambda + 1)^2(1 + 2K_1^2)^{-2}, \sigma^4\}$, then $h_\varsigma(t, \cdot) \geq 0$ in B . Taking $\varepsilon = \varepsilon_1 < \varepsilon_0$, it follows that $h_\varsigma \geq 0$ in $\Omega_{(s,T)}$ for any $\varsigma \in (0, 1)$. To complete the proof of the claim, we observe that

$$(\mathcal{B}\psi_\varsigma)(t, x) = \left(\frac{2\omega(x)\varepsilon_1}{t + \varsigma - s}\langle \beta(t, x), \nu(x) \rangle + \gamma(t, x)\right)\psi_\varsigma(t, x) > 0,$$

for any $(t, x) \in [s, T] \times \partial\Omega$ and $\varsigma \in (0, 1)$, since $\nabla\omega \equiv -\nu$ on $\partial\Omega$ and $\langle \beta, \nu \rangle \geq 0$ in $[s, T] \times \partial\Omega$ by Hypothesis 2.1(iii). The claim is now proved.

Finally, to check property (ii), we set $\Omega^{R,\eta} := \{x \in \Omega^R : \omega(x) \leq \eta\}$ and split $\Omega = \Omega^{R,\eta} \cup (\Omega \setminus \Omega^{R,\eta})$, where $\eta = \eta(\varsigma) > 0$ satisfies $\varsigma - \frac{\varepsilon\eta^2}{\varsigma} > 0$. It follows that $e^{\varsigma - \frac{\varepsilon\omega^2}{\varsigma}} > 1$ in $\Omega^{R,\eta}$ and, by continuity we can find $\tau = \tau(\varsigma) > 0$ such that

$$\varsigma^{\varepsilon\eta_0}\psi_\varsigma(t, x) = \frac{\varsigma^{\varepsilon\eta_0}e^{t+\varsigma-s}}{(t + \varsigma - s)^{\varepsilon\eta_0}}e^{-\frac{\varepsilon\omega^2(x)}{t+\varsigma-s}} > 1, \quad (t, x) \in [s, s + \tau] \times \Omega^{R,\eta}.$$

This implies that $w_{n,\varsigma} < v_n - M_n \leq 0$ in $([s, s + \tau] \times \Omega^{R,\eta}) \setminus S$. Moreover, since $w_{n,\varsigma}(s, \cdot) \leq 0$ in $\Omega^R \setminus \Omega^{R,\eta}$ and $w_{n,\varsigma}$ is continuous in $[s, T] \times \overline{\Omega^R \setminus \Omega^{R,\eta}}$, up to

replacing τ with a smaller value if needed, we can assume that $w_{n,\zeta}(t, x) \leq 0$ for any $(t, x) \in ([s, s + \tau] \times \Omega^R) \setminus S$. Property (ii) follows. \square

In order to get existence of a unique solution to the problem (P_B) we proceed by steps. In the following proposition we consider the case when the datum f vanishes at infinity and on the boundary of Ω . We recall that c_0 is the infimum of the potential c (see Hypothesis 2.1(iii)).

Proposition 3.3. *For any $f \in C_0(\Omega)$, the Cauchy problem (P_B) admits a unique bounded classical solution u . It belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s, +\infty)}}) \cap C_b(\overline{\Omega_{(s, +\infty)}})$ and satisfies the estimate*

$$\|u(t, \cdot)\|_\infty \leq e^{-c_0(t-s)} \|f\|_\infty, \quad t \geq s. \quad (3.3)$$

If, further, $f \in C_c^{2+\alpha}(\Omega)$, then $u \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s, +\infty)}})$.

Proof. Uniqueness follows immediately by applying Proposition 3.2. Estimate (3.3) can be obtained applying the same proposition to the functions $\pm e^{c_0(t-s)} u(t, x) - \|f\|_\infty$ which satisfy (3.1) with $t = +\infty$ and \mathcal{A} being replaced by $\mathcal{A} + c_0$.

To prove the existence part, we first consider $f \in C_c^{2+\alpha}(\Omega)$ and use an approximation argument. For any $n \in \mathbb{N}$, let ϑ_n be any smooth function such that $\chi_{B_n} \leq \vartheta_n \leq \chi_{B_{2n}}$. Moreover, let the functions $\mu_i \in C_b^{1+\alpha}(\Omega)$ ($i = 1, \dots, d$) satisfy $\mu_i = -D_i r_\Omega$ in a neighborhood of $\partial\Omega$. We then approximate the coefficients q_{ij} , b_j , c , β_i and γ ($i, j = 1, \dots, d$) by the (bounded) coefficients $q_{ij}^{(n)}$, $b_j^{(n)}$, $c^{(n)}$, $\beta_i^{(n)}$ and $\gamma^{(n)}$, defined by $q_{ij}^{(n)} = \vartheta_n q_{ij} + (1 - \vartheta_n) \delta_{ij}$, $b_j^{(n)} = \vartheta_n b_j$, $c^{(n)} = \vartheta_n c$, $\beta_i^{(n)} = \vartheta_n \beta_i + (1 - \vartheta_n) \mu_i$, and $\gamma^{(n)} = \vartheta_n \gamma$, for any $i, j = 1, \dots, d$. Clearly, $q_{ij}^{(n)}$, $b_j^{(n)}$, $c^{(n)}$ converge to q_{ij} , b_j , c , respectively, locally uniformly in $\overline{\Omega_I}$.

Let $\mathcal{A}^{(n)}$ and $\mathcal{B}^{(n)}$ be, respectively, the differential operators defined as \mathcal{A} and \mathcal{B} with $(q_{ij}, b_j, c, \beta, \gamma)$ being replaced by $(q_{ij}^{(n)}, b_j^{(n)}, c^{(n)}, \beta^{(n)}, \gamma^{(n)})$.

By [11, Thms. IV.5.2 & IV.5.3], for any $n \in \mathbb{N}$, the Cauchy problem

$$\begin{cases} D_t v(t, x) = (\mathcal{A}^{(n)} v)(t, x), & (t, x) \in (s, +\infty) \times \Omega, \\ (\mathcal{B}^{(n)} v)(t, x) = 0, & (t, x) \in (s, +\infty) \times \partial\Omega, \\ v(s, x) = f(x), & x \in \Omega, \end{cases} \quad (3.4)$$

admits a unique solution $u_n \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s, +\infty)}})$. Moreover, for any $m, n \in \mathbb{N}$, with $n > m$, the local Schauder estimates (see [11, Thm. IV.10.1]) show that there exists a positive constant c_m , independent of n , such that $\|u_n\|_{C^{1+\alpha/2, 2+\alpha}(\Omega_{(s, s+m)}^m)} \leq c_m \|f\|_{C_b^{2+\alpha}(\Omega)}$. Applying Arzelà-Ascoli theorem, we can determine a subsequence $(u_n^{(m)})$ converging in $C^{1,2}(\overline{\Omega_{(s, s+m)}^m})$ to a function $u^{(m)} \in C^{1+\alpha/2, 2+\alpha}(\Omega_{(s, s+m)}^m)$ which satisfies the equation $D_t u^{(m)} = \mathcal{A} u^{(m)}$ in $\Omega_{(s, s+m)}^m$. Moreover, $u^{(m)}(s, \cdot) \equiv f$ in Ω^m and $\mathcal{B}^{(n)} u^{(m)} \equiv \mathcal{B} u^{(m)} \equiv 0$ in $(s, s+m) \times (\partial\Omega \cap B_m)$. Since, without loss of generality, we can assume that $(u_n^{(m)}) \subset (u_n^{(m-1)})$ for any $m \in \mathbb{N}$, we can define the function $u : \overline{\Omega_{(s, +\infty)}} \rightarrow \mathbb{R}$ by setting $u(t, x) = u^{(m)}(t, x)$ for every $(t, x) \in \Omega_{(s, s+m)}^m$ and every $m \in \mathbb{N}$; it belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s, +\infty)}})$ and satisfies (P_B) .

Finally, we consider the general case when $f \in C_0(\Omega)$. We fix a sequence $(f_n) \subset C_c^{2+\alpha}(\Omega)$ converging to f uniformly in $\overline{\Omega}$ and denote by u_n the unique bounded classical solution to the Cauchy problem (P_B) , with f being replaced by f_n . Applying estimate (3.3) to the function $u_n - u_m$, we obtain that (u_n) is a Cauchy sequence in $\Omega_{(s, T)}$ for any $T > s$. Hence, by the arbitrariness of $T > s$, u_n converges uniformly in $\overline{\Omega_{(s, +\infty)}}$ to a function $u \in C_b(\overline{\Omega_{(s, +\infty)}})$ which satisfies $u(s, \cdot) = f$.

To prove that u is smooth, solves the differential equation and satisfies the boundary condition in (1.1), we apply a compactness argument, as in the first part of the proof, starting from the interior Schauder estimates (see e.g., [11, Thm. IV.10.1]) which show that the $C^{1+\alpha/2, 2+\alpha}$ -norm of the sequence (u_n) , in any compact set of $(s, T] \times \overline{\Omega}$, is bounded by a constant independent of n . Hence, there exists a subsequence (u_{n_k}) which converges locally uniformly, together with its derivatives, to a function $v \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega}_{(s, +\infty)})$ which satisfies the differential equation $D_t v = \mathcal{A}v$ in $\Omega_{(s, +\infty)}$ and the boundary condition $\mathcal{B}v = 0$ in $(s, +\infty) \times \partial\Omega$. Since, clearly, $v \equiv u$, u is the bounded classical solution to the problem $(P_{\mathcal{B}})$. \square

We can now address the general case when $f \in C_b(\Omega)$.

Theorem 3.4. *For any $f \in C_b(\Omega)$, problem $(P_{\mathcal{B}})$ has a unique bounded classical solution u . Moreover, $u \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega}_{(s, +\infty)})$, estimate (3.3) holds and, if $f \geq 0$ does not identically vanish then, u is strictly positive in $\Omega_{(s, +\infty)}$.*

Proof. The uniqueness part and estimate (3.3) are consequences of Proposition 3.2.

Let us prove the existence part. We fix $f \in C_b(\Omega)$ and a sequence $(f_n) \subset C_c^{2+\alpha}(\Omega)$ converging to f uniformly on compact subsets of Ω and such that $\|f_n\|_{\infty} \leq \|f\|_{\infty}$. We denote by u_n the bounded classical solution to problem $(P_{\mathcal{B}})$ with f being replaced by f_n . Using the interior Schauder estimates as in the last part of the proof of Proposition 3.3, we can prove that, up to a subsequence, u_n converges locally uniformly, together with its derivatives, to a function $u \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega}_{(s, +\infty)})$ which satisfies the differential equation $D_t u = \mathcal{A}u$ in $\Omega_{(s, +\infty)}$ and the boundary condition in $(s, +\infty) \times \partial\Omega$. So, to prove that u solves problem $(P_{\mathcal{B}})$ we have to show that u can be extended by continuity up to $t = s$ where it equals f . We fix a compact set $K \subset \Omega$ and a smooth and compactly supported function ψ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in K . Since ψf_n and $(1-\psi)f_n$ are compactly supported in Ω for every $n \in \mathbb{N}$, by linearity and Proposition 3.2 we conclude that $u_{f_n} = u_{\psi f_n} + u_{(1-\psi)f_n}$, where u_g denotes the unique bounded classical solution to problem $(P_{\mathcal{B}})$ with $f = g$. Applying Proposition 3.2 to the functions $\pm u_{(1-\psi)f_n} - \|f\|_{\infty}(1-u_{\psi})$, we get the estimate $\|u_{(1-\psi)f_n}\|_{\infty} \leq \|f\|_{\infty}(1-u_{\psi})$. Hence, $|u_{f_n} - f| \leq |u_{\psi f_n} - \psi f| + \|f\|_{\infty}(1-u_{\psi})$ in $\Omega_{(s, +\infty)}$. Letting $n \rightarrow +\infty$ in the previous inequality, from the last part of the proof of Proposition 3.3 we obtain $|u - f| \leq |u_{\psi f} - \psi f| + \|f\|_{\infty}(1-u_{\psi})$ in $\Omega_{(s, +\infty)}$. Since $\psi \equiv 1$ in K , it now follows that u can be extended by continuity at $t = s$ by setting $u(s, \cdot) = f$ in K . By the arbitrariness of K we deduce that $u = u_f$.

Finally let us prove that u is positive in $\Omega_{(s, +\infty)}$ if $f \geq 0$ is not identically zero. Proposition 3.2 shows that $u \geq 0$ in $\Omega_{(s, +\infty)}$. By contradiction, let us assume that there exists $(t_0, x_0) \in \Omega_{(s, +\infty)}$ such that $u(t_0, x_0) = 0$. Let us consider an open set $\Omega^* \ni x_0$, compactly contained in Ω , where f does not identically vanish. By applying [14, Thm. 3.7] to $-u$ in the cylinder $\Omega_{(s, t_0)}^*$ we deduce that $u(t, x) = 0$ for $(t, x) \in \times \Omega_{[s, t_0]}^*$ getting to a contradiction. \square

Remark 3.5. If the coefficients of the first-order operator \mathcal{B} are independent of t and belong to $C_b^{1+\alpha}(\partial\Omega)$, then, for any $f \in C_b(\overline{\Omega})$, the classical solution u_f to problem $(P_{\mathcal{B}})$ is continuous in the whole of $\overline{\Omega_{(s, +\infty)}}$. Indeed, under these conditions, we can repeat the same arguments as in the proof of Proposition 3.3 (without approximating the boundary operator \mathcal{B}) to show that, for any $g \in C_b^{2+\alpha}(\Omega)$ such that $\mathcal{B}g \equiv 0$ on $\partial\Omega$, the solution u_g to problem $(P_{\mathcal{B}})$ belongs to $C_{\text{loc}}^{(1+\alpha)/2, 2+\alpha}(\overline{\Omega_{(s, T)}})$. Since any function $f \in C_b(\overline{\Omega})$, which vanishes at ∞ , is the uniform limit of a sequence of functions (g_n) in $C_b^{2+\alpha}(\Omega)$, which vanish at infinity and satisfy $\mathcal{B}g_n \equiv 0$ on $\partial\Omega$ for any $n \in \mathbb{N}$, arguing as in the proof of Proposition 3.3, we conclude that u_f is continuous in $\overline{\Omega_{(s, +\infty)}}$ for any g as above. As a by product, in the proof of

Theorem 3.4 we can take as K a compact subset of $\overline{\Omega}$. Since $u_{f_n\psi}$ converges to $u_{f\psi}$ uniformly in $\Omega_{(s,+\infty)}$, we can estimate $|u_f - f| \leq |u_{\psi f} - \psi f| + \|f\|_\infty(1 - u_\psi)$. This inequality shows that u_f is continuous on $\{s\} \times K$. The arbitrariness of K yields the continuity of u_f on $\{s\} \times \overline{\Omega}$ and, consequently, in $\overline{\Omega_{(s,+\infty)}}$.

3.2. The general case. We now consider the general case when γ can assume also negative values. We stress that the arguments used in the previous subsection to prove uniqueness and estimate (3.3) fail. To overcome these difficulties we assume an additional assumption.

Hypothesis 3.6. *There exist a function $\phi \in C_{\text{loc}}^{2+\alpha}(\overline{\Omega}) \cap C_b(\overline{\Omega})$, with positive infimum, and a constant H such that $\mathcal{A}\phi \leq H\phi$ in Ω_I and $\mathcal{B}\phi \geq 0$ in $I \times \partial\Omega$.*

Theorem 3.7. *Let Hypotheses 3.6 be satisfied and fix $s \in I$. Then, for any $f \in C_b(\Omega)$, the problem $(P_{\mathcal{B}})$ admits a unique bounded classical solution u . The function u belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s,+\infty)}})$ and*

$$\|u(t, \cdot)\|_\infty \leq Me^{H(t-s)}\|f\|_\infty, \quad t > s, \quad (3.5)$$

where $M = (\inf_\Omega \phi)^{-1}\|\phi\|_\infty$. Finally, for any $f \in C_c^{2+\alpha}(\Omega)$, the unique bounded classical solution to problem $(P_{\mathcal{B}})$ belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s,+\infty)}})$ and, if $f \geq 0$ does not identically vanish, then $u > 0$ in $\Omega_{(s,+\infty)}$.

Proof. Let us fix a function $f \in C_b(\Omega)$. We point out that u is a bounded classical solution to problem $(P_{\mathcal{B}})$ if and only if the function $v : \overline{\Omega_{(s,+\infty)}} \rightarrow \mathbb{R}$, defined by $v(t, x) := e^{-H(t-s)}(\phi(x))^{-1}u(t, x)$ for any $(t, x) \in \overline{\Omega_{(s,+\infty)}}$, is a bounded classical solution to the Cauchy problem

$$\begin{cases} D_t v(t, x) = (\tilde{\mathcal{A}}v)(t, x), & (t, x) \in (s, +\infty) \times \Omega, \\ (\tilde{\mathcal{B}}v)(t, x) = 0, & (t, x) \in (s, +\infty) \times \partial\Omega, \\ v(s, x) = (\phi(x))^{-1}f(x), & x \in \Omega, \end{cases} \quad (3.6)$$

where

$$\tilde{\mathcal{A}}v = (\mathcal{A} + c)v + \frac{2}{\phi}\langle Q\nabla\phi, \nabla_x v \rangle - \left(H - \frac{\mathcal{A}\phi}{\phi}\right)v, \quad \tilde{\mathcal{B}}v = \langle \beta, \nabla_x v \rangle + \frac{\mathcal{B}\phi}{\phi}v.$$

Clearly, the coefficients of operators $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ satisfy Hypotheses 2.1 (note that the potential of the operator $\tilde{\mathcal{A}}$ is nonnegative in Ω_I). Moreover, Hypotheses 2.3 are satisfied as well and, by Hypotheses 3.6, it follows that $(\mathcal{B}\phi)/\phi \geq 0$ in $(s, +\infty) \times \partial\Omega$. Finally, we note that, for any bounded interval $J \subset I$, the function $\tilde{\varphi}_J$, defined by $\tilde{\varphi}_J(t, x) = e^{-H(t-s)}(\phi(x))^{-1}\varphi_J(t, x)$ for any $(t, x) \in \overline{\Omega_J}$, satisfies Hypothesis 2.4 with the same λ and the operators \mathcal{A} and \mathcal{B} being replaced by $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$. We can thus apply the results in Subsection 3.1 and deduce that the problem (3.6) admits a unique bounded classical solution v , which in addition belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s,+\infty)}})$ and satisfies the estimate $\|v(t, \cdot)\|_\infty \leq e^{H(t-s)}\|f/\phi\|_\infty$ for any $t \geq s$. As a byproduct we deduce that problem $(P_{\mathcal{B}})$ admits a unique bounded classical solution which, in addition, belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s,+\infty)}})$ and satisfies the inequality (3.5).

The last assertions follow from Proposition 3.3 and Theorem 3.4, observing that the operator $f \mapsto f/\phi$ preserves positivity and it is an isomorphism from $C_c^{2+\alpha}(\Omega)$ into $C_c^{2+\alpha}(\Omega)$ and from $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\overline{\Omega_{(s,+\infty)}})$ into itself. \square

Remark 3.8. From the proof of Theorem 3.7 and Remark 3.5 it follows that, if the coefficients of the operator \mathcal{B} are independent of t and belong to $C_b^{1+\alpha}(\overline{\Omega})$ as well as the function $(\mathcal{B}\phi)/\phi$, then the bounded classical solution u_f to problem $(P_{\mathcal{B}})$ is continuous in $\overline{\Omega_{(s,+\infty)}}$ for any $f \in C_b(\overline{\Omega})$.

4. THE EVOLUTION OPERATOR: CONTINUITY PROPERTIES AND COMPACTNESS

Set $\Lambda = \{(t, s) \in I \times I : t > s\}$. In view of Theorems 3.4 and 3.7, the family of bounded linear operators $\{G_{\mathcal{B}}(t, s) : (t, s) \in \bar{\Lambda}\}$, defined in $C_b(\Omega)$ by $G_{\mathcal{B}}(t, t) = id_{C_b(\Omega)}$ and $G_{\mathcal{B}}(t, s)f := u_f(t, \cdot)$ for $t > s$, where u_f is the unique solution to the problem $(P_{\mathcal{B}})$ with $f \in C_b(\Omega)$, gives rise to an evolution operator. Each operator $G_{\mathcal{B}}(t, s)$ is positive. Moreover, estimates (3.3) and (3.5) imply that

$$\|G_{\mathcal{B}}(t, s)f\|_{\infty} \leq \begin{cases} e^{-c_0(t-s)}\|f\|_{\infty}, & \gamma \geq 0, \\ Me^{H(t-s)}\|f\|_{\infty}, & \text{otherwise,} \end{cases} \quad t > s, f \in C_b(\Omega). \quad (4.1)$$

In particular, if $\gamma \geq 0$, $G_{\mathcal{B}}(t, s)$ is a contraction. On the other hand, for a general γ , the proof of Theorem 3.7 shows that

$$G_{\mathcal{B}}(t, s)f = \phi e^{H(t-s)} \tilde{G}_{\mathcal{B}}(t, s) \left(\frac{f}{\phi} \right), \quad t > s \in I, x \in \Omega, f \in C_b(\Omega), \quad (4.2)$$

where $\tilde{G}_{\mathcal{B}}(t, s)$ is the evolution operator associated to problem (3.6). In what follows, to simplify the notation we denote the evolution operator $\{G_{\mathcal{B}}(t, s) : (t, s) \in \bar{\Lambda}\}$ simply by $G_{\mathcal{B}}(t, s)$.

We now prove some continuity property of the evolution operator $G_{\mathcal{B}}(t, s)$, which will be used in Section 5.

Proposition 4.1. *Let $(f_n) \subset C_b(\Omega)$ be a bounded sequence with respect to the sup-norm and let $f \in C_b(\Omega)$.*

- (i) *If f_n converges pointwise to f in Ω , then, for any pair of compact sets $J \subset (s, +\infty)$ and $K \subset \Omega$, $G_{\mathcal{B}}(\cdot, s)f_n$ converges to $G_{\mathcal{B}}(\cdot, s)f$ in $C^{1,2}(J \times K)$ as $n \rightarrow +\infty$.*
- (ii) *If f_n converges locally uniformly to f in Ω , then $G_{\mathcal{B}}(\cdot, s)f_n$ converges uniformly to $G_{\mathcal{B}}(\cdot, s)f$ in $[s, T] \times K$ for any $T > s$ and any compact set $K \subset \Omega$.*

Proof. (i) Since $\sup_{n \in \mathbb{N}} \|f_n\|_{\infty} < +\infty$, estimate (4.1) and the interior Schauder estimates show that the sequence $(G_{\mathcal{B}}(\cdot, s)f_n)$ is bounded in $C^{1+\alpha/2, 2+\alpha}(J \times K)$ for any J and K as in the statement. The compactness argument already used in the proof of Theorem 3.4 shows that, there exists a subsequence $(G_{\mathcal{B}}(\cdot, s)f_{n_k})$ which converges in $C^{1,2}(J \times K)$ to some function v . To infer that $v = G_{\mathcal{B}}(\cdot, s)f$, we observe that, for any $(t, s, x) \in \Lambda \times \Omega$ the map $f \mapsto (G_{\mathcal{B}}(t, s)f)(x)$ defines a positive and bounded operator from $C_0(\Omega)$ to \mathbb{R} . The Riesz representation theorem shows that there exists a family of positive and finite Borel measures $\{g(t, s, x, dy) : (t, s, x) \in \Lambda \times \Omega\}$ such that

$$(G_{\mathcal{B}}(t, s)\psi)(x) = \int_{\Omega} \psi(y)g(t, s, x, dy), \quad I \ni s < t, x \in \Omega, \psi \in C_0(\Omega). \quad (4.3)$$

From the proof of Theorem 3.4 we know that, if $(\psi_n) \subset C_c^{2+\alpha}(\Omega)$ is bounded with respect to the sup-norm and converges to some $\psi \in C_b(\Omega)$, locally uniformly in Ω , then $G_{\mathcal{B}}(t, s)\psi_n$ converges to $G_{\mathcal{B}}(t, s)\psi$ locally uniformly in Ω as well. Hence, (4.3) can be extended to any $\psi \in C_b(\Omega)$. Writing it with ψ being replaced by f_n and letting $n \rightarrow +\infty$, by dominated convergence we conclude that $v = G_{\mathcal{B}}(\cdot, s)f$. Since the limit is independent of the sequence (n_k) , the whole sequence $(G_{\mathcal{B}}(\cdot, s)f_n)$ converges to $G_{\mathcal{B}}(\cdot, s)f$ in $C^{1,2}(J \times K)$.

(ii) In view of formula (4.2), we can limit ourselves to dealing with the case when $\gamma \geq 0$. For this purpose, fix $T > s$, a compact set $K \subset \Omega$ and $\varepsilon > 0$. Further, let $\eta \in C_c^{\infty}(\Omega)$ be such that $\eta \equiv 1$ in K . We can split $f_n = \eta f_n + (1 - \eta)f_n$ for any $n \in \mathbb{N}$. Then, by linearity $G_{\mathcal{B}}(\cdot, s)f_n = G_{\mathcal{B}}(\cdot, s)(\eta f_n) + G_{\mathcal{B}}(\cdot, s)((1 - \eta)f_n)$. Since $\eta f_n \in C_c(\Omega)$ and converges uniformly to ηf , by (4.1) we deduce that $G_{\mathcal{B}}(\cdot, s)(\eta f_n)$ converges to $G_{\mathcal{B}}(\cdot, s)(\eta f)$ uniformly in $\Omega_{[s, T]}$. Hence, we just need to prove that

$G_{\mathcal{B}}(\cdot, s)((1 - \eta)f_n)$ converges to $G_{\mathcal{B}}(\cdot, s)((1 - \eta)f_n)$ locally uniformly in $[s, T] \times K$. The arguments in the proof of Theorem 3.4 show that

$$|G_{\mathcal{B}}(t, s)((1 - \eta)f_n)(x)| \leq \sup_{n \in \mathbb{N}} \|f_n\|_{\infty} [1 - (G_{\mathcal{B}}(t, s)\eta)(x)], \quad t > s, x \in \overline{\Omega}, n \in \mathbb{N}.$$

Therefore, as $t \rightarrow s^+$, $G_{\mathcal{B}}(t, s)((1 - \eta)f_n)$ and $G_{\mathcal{B}}(t, s)((1 - \eta)f)$ converge to 0 uniformly in K and uniformly with respect to $n \in \mathbb{N}$. Hence, we can determine $\delta \in (0, T - s)$ such that

$$\|G_{\mathcal{B}}(\cdot, s)((1 - \eta)f_n)\|_{C([s, s+\delta] \times K)} + \|G_{\mathcal{B}}(\cdot, s)((1 - \eta)f)\|_{C([s, s+\delta] \times K)} \leq \frac{1}{2}\varepsilon. \quad (4.4)$$

On the other hand, property (i) implies that $G_{\mathcal{B}}(\cdot, s)((1 - \eta)f_n)$ converges to $G_{\mathcal{B}}(\cdot, s)((1 - \eta)f)$ uniformly in $[s + \delta, T] \times K$. Thus, there exists $n_0 \in \mathbb{N}$ such that $\|G_{\mathcal{B}}(\cdot, s)((1 - \eta)f_n) - G_{\mathcal{B}}(\cdot, s)((1 - \eta)f)\|_{C([s+\delta, T] \times K)} \leq \varepsilon/2$ for any $n \geq n_0$. From this estimate and (4.4) we conclude that $G_{\mathcal{B}}(\cdot, s)((1 - \eta)f_n)$ tends to $G_{\mathcal{B}}(\cdot, s)((1 - \eta)f)$ uniformly in $[s, T] \times K$. \square

Remark 4.2. The proof of Proposition 4.1 shows that

$$(G_{\mathcal{B}}(t, s)f) = \int_{\Omega} f(y)g(t, s, x, dy), \quad t > s \in I, x \in \Omega \quad (4.5)$$

for any $f \in C_b(\Omega)$. Since any Borel bounded function $f : \Omega \rightarrow \mathbb{R}$ can be approximated pointwise by a bounded sequence in $C_b(\Omega)$, the dominated convergence theorem allows to extend the evolution operator $G_{\mathcal{B}}(t, s)$ to all the bounded and Borel measurable functions, via formula (4.5). Moreover, the same arguments in the proof of Proposition 4.1(i) show that $G_{\mathcal{B}}(\cdot, s)f \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\Omega_{(s, +\infty)})$ for any bounded and Borel measurable function f . In particular, this shows that $G(t, s)$ is Strong Feller.

In the following proposition, under an additional smoothness assumption on the coefficients of the operator \mathcal{A} , we show that each measure $g(t, s, x, dy)$ is absolutely continuous with respect the Lebesgue measure, and we prove some smoothness properties of its density.

Proposition 4.3. *Assume that $q_{ij} \in L_{\text{loc}}^{\infty}(I; W_{\text{loc}}^{1,p}(\Omega))$ for some $p > d + 2$. Then, there exists a unique Green function $g_{\mathcal{B}} : \Lambda \times \Omega \times \Omega \rightarrow (0, +\infty)$ associated with the Cauchy problem $(P_{\mathcal{B}})$, i.e.,*

$$(G_{\mathcal{B}}(t, s)f)(x) = \int_{\Omega} g_{\mathcal{B}}(t, s, x, y)f(y) dy, \quad s < t, x \in \Omega, \quad (4.6)$$

for every $f \in C_b(\Omega)$. Moreover, for any $s \in I$ and $y \in \Omega$, the function $g_{\mathcal{B}}(\cdot, s, \cdot, y)$ belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\Omega_{(s, +\infty)})$ and satisfies $D_t g_{\mathcal{B}} - \mathcal{A} g_{\mathcal{B}} \equiv 0$ in $\Omega_{(s, +\infty)}$. Finally, $g_{\mathcal{B}}(t, s, x, \cdot) \in L^1(\Omega)$ for every $(t, s) \in \Lambda$, $x \in \Omega$ and

$$\|g_{\mathcal{B}}(t, s, x, \cdot)\|_{L^1(\Omega)} \leq \begin{cases} e^{-c_0(t-s)}, & \gamma \geq 0, \\ Me^{H(t-s)}, & \text{otherwise,} \end{cases} \quad (t, s) \in \Lambda, x \in \Omega, \quad (4.7)$$

where H and M are as in (3.5).

Proof. We split the proof into four steps. In the first three steps we consider the case when $\gamma \geq 0$. In the last one, we address the general case.

Step 1. First, we prove that there exists a function $g_{\mathcal{B}} : \Lambda \times \Omega \times \Omega \rightarrow (0, +\infty)$ such that (4.6) holds. For this purpose, we consider the evolution operator $G_n(t, s)$ associated to the Cauchy problem (3.4) in $C_b(\Omega)$. It is well known (see...) that

for every $n \in \mathbb{N}$ there exists a function $g_n : \Lambda \times \Omega \times \Omega \rightarrow \mathbb{R}^+$ such that, for any $I \ni s < t$ and $x \in \Omega$, the function $g_n(t, s, x, \cdot)$ belongs to $L^1(\Omega)$ and

$$(G_n(t, s)f)(x) = \int_{\Omega} f(y)g_n(t, s, x, y)dy, \quad f \in C_b(\Omega), \quad n \in \mathbb{N}. \quad (4.8)$$

Moreover, the function $g_n(\cdot, s, \cdot, y)$ belongs to $C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\Omega_{(s, +\infty)})$ for any $s \in I$, $y \in \Omega$ and $D_t g_n(\cdot, s, \cdot, y) = \mathcal{A}^{(n)} g_n(\cdot, s, \cdot, y)$ in $\Omega_{(s, +\infty)}$. Finally, the maximum principle in Proposition 3.2 immediately implies that $\|g_n(t, s, x, \cdot)\|_{L^1(\Omega)} \leq 1$ for any $t > s \in I$ and $x \in \Omega$.

We now fix $t \in I$, $s_1, s_2 \in (-\infty, t) \cap I$ and $x \in \Omega$. Formula (A.1) shows that $D_s((G_n(t, s)\psi(s, \cdot))(x)) = (G_n(t, s)D_s\psi(s, \cdot))(x) - (G_n(t, s)\mathcal{A}^{(n)}(s)\psi(s, \cdot))(x)$, for any $(s, x) \in \Omega_{I \cap (-\infty, t]}$ and any $\psi \in C_c^{1,2}(\Omega_{(s_1, s_2)})$. Hence, taking (4.8) into account we deduce that

$$\int_{\Omega_{(s_1, s_2)}} (D_r\psi(r, y) - \mathcal{A}^{(n)}(r)\psi(r, y))g_n(t, r, x, y)dr dy = 0, \quad (4.9)$$

since $\psi(s_1, \cdot) = \psi(s_2, \cdot) \equiv 0$. Applying [7, Cor. 3.9] to the measures $\mu_n(dr, dy) = g_n(t, r, x, y)dydr$, we deduce that $g_n(t, \cdot, x, \cdot)$ is locally θ -Hölder continuous in $\Omega_{(s_1, s_2)}$ for some $\theta \in (0, 1)$, and, by the arbitrariness of $s_1 < s_2 < t$, $g_n(t, \cdot, x, \cdot) \in C_{\text{loc}}^{\theta}(\Omega_{(-\infty, t) \cap I})$ for any $(t, x) \in \Omega_I$. Moreover, an inspection of the proof of [7, Thm 3.8], shows that the C^{θ} -norm of $g_n(t, \cdot, x, \cdot)$ over any compact set $[a, b] \times K \subset \Omega_{(-\infty, t) \cap I}$ can be bounded from above by a constant, independent of n . Hence, a straightforward compactness argument allows to prove that, for any fixed $(t, x) \in \Omega_{(s, +\infty)}$, there exist a subsequence $(n_k) \subset \mathbb{N}$ and a function $g_{t,x} \in C_{\text{loc}}^{\theta}(\Omega_{(-\infty, t) \cap I})$ such that $g_n(t, \cdot, x, \cdot)$ converges to $g_{t,x}$ locally uniformly in $\Omega_{(-\infty, t) \cap I}$. Moreover, since $\|g_n(t, s, x, \cdot)\|_{L^1(\Omega)} \leq 1$ for any $n \in \mathbb{N}$, Fatou lemma shows that $g_{t,x}(s, \cdot) \in L^1(\Omega)$ for any $I \ni s < t$. Hence, we can write (4.8), with n being replaced by n_k , and let $k \rightarrow +\infty$, using the dominated convergence theorem and taking Theorem 3.4 into account, to get

$$(G_{\mathcal{B}}(t, s)f)(x) = \int_{\Omega} f(y)g_{t,x}(s, y) dy, \quad f \in C_b(\Omega). \quad (4.10)$$

Formula (4.10) shows also that the function $g_{t,x}$ does not depend on subsequence (n_k) and, therefore all the sequence $(g_n(t, \cdot, x, \cdot))$ converge to $g_{t,x}$ locally uniformly in $\Omega_{(-\infty, t) \cap I}$. Formula (4.6) follows with $g_{\mathcal{B}}(t, s, x, y) = g_{t,x}(s, y)$. Moreover, estimate (4.1) and formula (4.10) lead to (4.7).

Step 2. Let us now prove that $g_{\mathcal{B}}(\cdot, s, \cdot, y) \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}(\Omega_{(s, +\infty)})$ and solves $D_t g_{\mathcal{B}}(\cdot, s, \cdot, y) - \mathcal{A} g_{\mathcal{B}}(\cdot, s, \cdot, y) = 0$ in $\Omega_{(s, +\infty)}$. For this purpose, we begin by observing that, since $D_t g_n(\cdot, s, \cdot, y) - \mathcal{A}^{(n)} g_n(\cdot, s, \cdot, y) = 0$ in $\Omega_{(s, +\infty)}$, the classical interior Schauder estimates yield that

$$\|g_n(\cdot, s, \cdot, y)\|_{C^{1+\alpha/2, 2+\alpha}(\Omega'_{(T_1, T_2)})} \leq c \|g_n(\cdot, s, \cdot, y)\|_{L^{\infty}(\Omega''_{(T_0, T_2)})}, \quad (4.11)$$

for any $s < T_0 < T_1 < T_2$, any $\Omega' \Subset \Omega'' \Subset \Omega$ and some positive constant c independent of n , since the coefficients of the operator $\mathcal{A}^{(n)}$ converge to the coefficients of the operator \mathcal{A} , locally uniformly in $\Omega_{(s, +\infty)}$.

We claim that, for every $s \in I$ and $y \in \Omega$, the function $g_n(\cdot, s, \cdot, y)$ is bounded in $\Omega''_{(T_0, T_2)}$, uniformly with respect to n . So, let us fix $s \in I$, $y \in \Omega$ and denote by (t_h) and (x_k) two countable sets dense in $[T_0, T_2 + 1]$ and in Ω'' , respectively. By Step 1, $(g_n(t_h, s, x_k, \cdot))$ converges locally uniformly in Ω to $g_{\mathcal{B}}(t_h, s, x_k, \cdot)$ for any $h, k \in \mathbb{N}$, as $n \rightarrow +\infty$. In particular, there exists a positive constant $c(t_h, x_k)$ such that $g_n(t_h, s, x_k, y) \leq c(t_h, x_k)$ for every $h, k, n \in \mathbb{N}$.

Let $R > 1$ be such that $s < T_0 - 2/R$ and let $2r := \text{dist}(\Omega'', \partial\Omega)$. Since $D_t g_n(\cdot, s, \cdot, y) - \mathcal{A}^{(n)} g_n(\cdot, s, \cdot, y) = 0$ in $[T_0 - 1/R, T_2 + 2] \times \bigcup_{x \in \Omega''} B_r(x)$, using

the arguments in [7, Cor. 3.11] based on the Harnack inequality in [4, Thm. 3], we see that, if $\rho^2 < \min\{r^2, R^{-1}\}$, then there exists a positive constant M_0 , independent of n , such that $\max_{\overline{W_1}} g_n(\cdot, s, \cdot, y) \leq M_0 \min_{\overline{W_2}} g_n(\cdot, s, \cdot, y)$, where $W_1 = (t_h - \frac{3}{4}\rho^2, t_h - \frac{1}{2}\rho^2) \times B_{\rho/2}(x_k)$ and $W_2 = (t_h - \frac{1}{4}\rho^2, t_h) \times B_{\rho/2}(x_k)$. Consequently, $g_n(t, s, x, y) \leq M_0 g_n(t_h, s, x_k, y) \leq M_0 c(t_h, x_k)$ for every $t \in [t_h - \frac{3}{4}\rho^2, t_h - \frac{1}{2}\rho^2]$, $x \in \overline{B_{\rho/2}(x_k)}$. Since $\Omega''_{[T_0, T_2]}$ can be covered by a finite number of cylinders $[t_h - \frac{3}{4}\rho^2, t_h - \frac{1}{2}\rho^2] \times B_{\rho/2}(x_k)$, from the previous chain of inequalities we deduce that $g_n(\cdot, s, \cdot, y)$ is uniformly bounded in $\Omega''_{[T_0, T_2]}$ by a constant independent of n , as it has been claimed.

From (4.11), we now deduce that for any s, y as above, the $C^{1+\alpha/2, 2+\alpha}$ -norm of the function $g_n(\cdot, s, \cdot, y)$ over the cylinder $\Omega'_{(T_1, T_2)}$ can be bounded from above by a constant independent of n . Consequently, by the Arzelà-Ascoli theorem, there exist an increasing sequence $(n_k) \subset \mathbb{N}$ and a function $\tilde{g}_{s,y} \in C^{1+\alpha/2, 2+\alpha}(\Omega'_{(T_1, T_2)})$ such that $g_{n_k}(\cdot, s, \cdot, y)$ converges to $\tilde{g}_{s,y}$ in $C^{1,2}(\Omega'_{(T_1, T_2)})$, as $k \rightarrow +\infty$. Clearly $\tilde{g}_{s,y}$ belongs to $C^{1+\alpha/2, 2+\alpha}(\Omega'_{(T_1, T_2)})$ and satisfies the differential equation in $(P_{\mathcal{B}})$ in $\Omega'_{(T_1, T_2)}$. Since g_n converges to $g_{\mathcal{B}}$ pointwise in $\Lambda \times \Omega \times \Omega$, we can infer that $\tilde{g}_{s,y} = g_{\mathcal{B}}(\cdot, s, \cdot, y)$. The arbitrariness of T_1, T_2, Ω' allows us to conclude that $g_{\mathcal{B}}(\cdot, s, \cdot, y) \in C^{1+\alpha/2, 2+\alpha}_{\text{loc}}(\Omega_{(s, +\infty)})$ and $D_t g_{\mathcal{B}}(\cdot, s, \cdot, y) - \mathcal{A} g_{\mathcal{B}}(\cdot, s, \cdot, y) = 0$ in $\Omega_{(s, +\infty)}$.

Step 3. Here, we prove that $g_{\mathcal{B}}$ is strictly positive in $\Lambda \times \Omega \times \Omega$. From Theorem 3.4 we know that $G_{\mathcal{B}}(t, s)f$ is strictly positive in Ω , for any $f \in C_b(\Omega)$. This implies that $g_{\mathcal{B}}$ is nonnegative. Indeed, if $g(t_0, s_0, x_0, y_0) < 0$ for some $(t_0, s_0, x_0, y_0) \in \Lambda \times \Omega \times \Omega$, since the function $g_{\mathcal{B}}(t_0, s_0, x_0, \cdot)$ is continuous in Ω , there would exist $\sigma > 0$ such that $g(t_0, s_0, x_0, \cdot) < 0$ in $B_{\sigma}(y_0)$. Taking a function $f \in C_b(\Omega)$ such that $\chi_{B_{\sigma/2}(y_0)} \leq f \leq \chi_{B_{\sigma}(y_0)}$ we would get $(G(t_0, s_0)f)(x_0) < 0$, which is a contradiction.

Suppose that $g_{\mathcal{B}}(t_0, s_0, x_0, y_0) = 0$ at some point (t_0, s_0, x_0, y_0) . Fix $t_1 < s_1 < s_2 < s_0 < t_2 < t_0$ and $R > 0$ such that $B_R(y_0) \Subset \Omega$. Letting $n \rightarrow +\infty$ in (4.9) we get that $(D_r - \mathcal{A})^* g_{\mathcal{B}}(t_0, \cdot, x_0, \cdot) = 0$ in $(t_1, t_2) \times B_R(y_0)$. Hence, using again the arguments in [7, Cor. 3.11] we deduce that $g_{\mathcal{B}}(t_0, \cdot, x_0, \cdot) = 0$ in $(s_1, s_2) \times B_R(y_0)$. Now, fix $s \in (s_1, s_2)$ and a nonnegative function $f \in C_c(B_R(y_0))$ such that $f(y_0) > 0$. From (4.6) it follows that $(G_{\mathcal{B}}(t_0, s)f)(x_0) = 0$, which cannot be the case since $G_{\mathcal{B}}(t_0, s)f > 0$ in Ω .

Step 4. Finally, in the general case when no assumption on the sign of γ is assumed, we can apply the first part of the proof to the evolution operator $\tilde{G}_{\tilde{\mathcal{B}}}(t, s)$ associated with the Cauchy problem (3.6) in $C_b(\Omega)$. Then, there exists a unique function $\tilde{g}_{\tilde{\mathcal{B}}} : \Lambda \times \Omega \times \Omega \rightarrow (0, +\infty)$ such that

$$(\tilde{G}_{\tilde{\mathcal{B}}}(t, s)f)(x) = \int_{\Omega} \tilde{g}_{\tilde{\mathcal{B}}}(t, s, x, y)f(y) dy, \quad s < t, \quad x \in \Omega, \quad f \in C_b(\Omega).$$

Moreover, for any $s \in I$ and any $y \in \Omega$, $\tilde{g}_{\tilde{\mathcal{B}}}(\cdot, s, \cdot, y) \in C^{1+\alpha/2, 2+\alpha}_{\text{loc}}(\Omega_{(s, +\infty)})$, satisfies $D_t \tilde{g}_{\tilde{\mathcal{B}}} - \tilde{\mathcal{A}} \tilde{g}_{\tilde{\mathcal{B}}} = 0$ in $\Omega_{(s, +\infty)}$ and $\|\tilde{g}_{\tilde{\mathcal{B}}}(t, s, x, \cdot)\|_{L^1(\Omega)} \leq 1$ for any $(t, s) \in \Lambda$, $x \in \Omega$. Now, taking into account formula (4.2) we conclude that the function $g_{\mathcal{B}}$, defined by $g_{\mathcal{B}}(t, s, x, y) = \phi(x)(\phi(y))^{-1} e^{H(t-s)} \tilde{g}_{\tilde{\mathcal{B}}}(t, s, x, y)$ for any $(t, s, x, y) \in \Lambda \times \Omega \times \Omega$, satisfies the claim and estimate (4.7) holds. \square

4.1. Compactness. We now provide a sufficient condition for $G_{\mathcal{B}}(t, s)$ to be compact in $C_b(\Omega)$. In the case when Ω is replaced by \mathbb{R}^d , the compactness of the evolution operator $G_{\mathcal{B}}(t, s)$ has been studied in [2, 13].

Remark 4.4. Suppose that $\gamma \geq 0$ and let $G_n(t, s)$ be the evolution operator associated with the pair $(\mathcal{A}^{(n)}, \mathcal{B}^{(n)})$ in $C_b(\Omega)$, introduced in the proof of Proposition 3.3. As it has been shown, for any $f \in C^{2+\alpha}_c(\Omega)$, $G_n(t, s)f$ converges to $G_{\mathcal{B}}(t, s)f$ locally uniformly in Ω , as $n \rightarrow +\infty$, for any $I \ni s < t$. Actually, this happens

also if $f \in C_b(\Omega)$. If $f \in C_c(\Omega)$, let the sequence $(f_k) \subset C_c^{2+\alpha}(\Omega)$ converge to f uniformly in Ω . Then, splitting $G_n(t, s)f - G_{\mathcal{B}}(t, s)f = (G_n(t, s)f - G_n(t, s)f_k) + (G_n(t, s)f_k - G_{\mathcal{B}}(t, s)f_k) + (G_{\mathcal{B}}(t, s)f_k - G_{\mathcal{B}}(t, s)f)$ and observing that, by the maximum principle in Proposition 3.2, $\|G_n(t, s)f\|_{\infty} \leq \|f\|_{\infty}$ for any $I \ni s < t$ and $f \in C_b(\Omega)$, we easily deduce that

$$\|G_n(t, s)f - G_{\mathcal{B}}(t, s)f\|_{C(K)} \leq 2\|f_k - f\|_{\infty} + \|G_n(t, s)f_k - G_{\mathcal{B}}(t, s)f_k\|_{C(K)},$$

for any compact set $K \subset \Omega$. Letting, first n and then k tend to $+\infty$, we deduce that $G_n(t, s)f$ tends to $G_{\mathcal{B}}(t, s)f$ locally uniformly in Ω .

In the general case, we can argue as in the proof of Theorem 3.4, replacing u_{f_n} with $G_n(\cdot, s)f$, $u_{(1-\psi)f_n}$ with $G_n(\cdot, s)((1-\psi)f)$ and $u_{\psi f_n}$ with $G_n(\cdot, s)(\psi f)$. Splitting $G_n(t, s)f = G_n(t, s)(\psi f) + G_n(t, s)((1-\psi)f)$ and observing that the estimate $|G_n(t, s)((1-\psi)f)| \leq \|f\|_{\infty}(1 - G_n(t, s)\psi)$ holds for any $t > s$, we get $|(G_n(t, s)f)(x) - f(x)| \leq |(G_n(t, s)(\psi f))(x) - f(x)\psi(x)| + \|f\|_{\infty}(1 - (G_n(t, s)\psi)(x))$ for any $(t, x) \in (s, +\infty) \times K$. Let $(G_{n_k}(\cdot, s)f)$ be a subsequence which converges, locally uniformly in $\Omega_{(s, +\infty)}$ to a function u , which turns out to solve the differential equation $D_t u - \mathcal{A}u = 0$ in $\Omega_{(s, +\infty)}$ and satisfies the boundary condition $Bu = 0$ on $(s, +\infty) \times \partial\Omega$. Since $G_n(\cdot, s)(\psi f)$ converges to $G_{\mathcal{B}}(\cdot, s)(\psi f)$ locally uniformly in $\Omega_{(s, +\infty)}$, we can write the previous estimate with n being replaced by n_k and let $k \rightarrow +\infty$, to infer that $|u(t, x) - f(x)| \leq |(G_{\mathcal{B}}(t, s)(\psi f))(x) - f(x)\psi(x)| + \|f\|_{\infty}(1 - (G_{\mathcal{B}}(t, s)\psi)(x))$ and conclude that u is continuous on $\{s\} \times K$. Hence, $u = G_{\mathcal{B}}(\cdot, s)f$. Since the limit is independent of the sequence (n_k) , the whole sequence $G_n(\cdot, s)f$ converges to $G_{\mathcal{B}}(\cdot, s)f$ locally uniformly in Ω .

Theorem 4.5. *Assume that there exist a bounded interval $J \subset I$, $c_1, c_2 > 0$ a positive function $\psi \in C^2(\Omega)$ and $\varepsilon > 0$, such that $\lim_{x \in \Omega, |x| \rightarrow \infty} \psi(x) = \infty$ and $(\mathcal{A}(t)\psi)(x) \leq c_2 - c_1(\psi(x))^{1+\varepsilon}$ for any $(t, x) \in J \times \Omega$. Then, $G_{\mathcal{B}}(t, s)$ is compact in $C_b(\Omega)$ for any $(t, s) \in \Lambda \cap J^2$.*

Proof. In view of formula (4.2), we can limit ourselves to considering the case when $\gamma \geq 0$. Moreover, it suffices to prove that

$$\lim_{n \rightarrow +\infty} \sup_{x \in \Omega} g_{\mathcal{B}}(t, s, x, \Omega \setminus \Omega_n) = 0, \quad (t, s) \in \Lambda \cap J^2, \quad (4.12)$$

where the measures $g_{\mathcal{B}}(t, s, x, dy)$ are defined in (4.5). Formula (4.12) implies that for any $(t, s) \in \Lambda \cap J^2$, the evolution operator $G_{\mathcal{B}}(t, s)$ is the limit of the sequence of operators (S_n) defined by $S_n f = G_{\mathcal{B}}(t, r)(\chi_{\Omega_n} G_{\mathcal{B}}(r, s)f)$ for any $f \in C_b(\Omega)$ and $n \in \mathbb{N}$, where $r = (s+t)/2$. Indeed, $\|G_{\mathcal{B}}(t, s) - S_n\|_{\mathcal{L}(C_b(\Omega))} \leq \sup_{x \in \Omega} g_{\mathcal{B}}(t, r, x, \Omega \setminus \Omega_n)$. Note that each operator S_n is well defined, in view of Remark 4.2, and is compact. Indeed, if (f_k) is a bounded sequence in $C_b(\Omega)$, then the interior Schauder estimates shows that the sequence $(G_{\mathcal{B}}(r, s)f_k)$ is bounded in $C^{2+\alpha}(\Omega_n)$. Hence, it admits a subsequence $(G_{\mathcal{B}}(r, s)f_{k_m})$ uniformly converging in Ω_n . As a byproduct, using formula (4.5), we deduce that $S_n f_{k_m}$ converges uniformly in Ω as $m \rightarrow +\infty$, whence S_n is a compact operator.

To prove (4.12) we argue as follows: first we prove that ψ is integrable with respect to the measures $g(t, s, x, dy)$ for any $(t, s) \in \Lambda \cap J$ and $x \in \Omega$ (in what follows, with a slight abuse we denote by $(G_{\mathcal{B}}(t, s)\psi)(x)$ the integral of the function ψ with respect to the measure $g(t, s, x, dy)$, whenever the integral makes sense, even if it is not finite). Then, we show that, for any $s \in J$ and $\delta > 0$, $(G_{\mathcal{B}}(t, s)\psi)(x)$ is bounded in $([s + \delta, +\infty) \cap J) \times \Omega$. This is enough for our aims. Indeed,

$$g(t, s, x, \Omega \setminus \Omega_n) \leq \frac{1}{k_n} \int_{\Omega \setminus \Omega_n} \psi(y) g(t, s, x, dy) \leq \frac{1}{k_n} (G_{\mathcal{B}}(t, s)\psi)(x) \leq \frac{M}{k_n},$$

for some positive constant M , where $k_n := \inf\{\psi(y) : y \in \Omega \setminus \Omega_n\}$ tends to $+\infty$ as $n \rightarrow +\infty$.

We split the rest of the proof in three steps.

Step 1. For any $n \in \mathbb{N}$, let $\psi_n = \phi_n \circ \psi$, where $\phi_n \in C^2([0, +\infty))$ is an increasing and concave function such that $\phi_n(r) = r$ for any $r \in (0, n)$ and $\phi_n(r) = n + 1$ for any $r \in (n + 2, +\infty)$. Clearly, for any $n \in \mathbb{N}$, ψ_n belongs to $C^2(\Omega)$ and is constant outside a compact set. Let us prove that $(G_{\mathcal{B}}(t, \sigma)\mathcal{A}(\sigma)\psi_n)(x)$ is well defined for any $(t, \sigma) \in \Lambda \cap J^2$, $x \in \Omega$ and

$$(G_{\mathcal{B}}(t, r)\psi_n)(x) - (G_{\mathcal{B}}(t, s)\psi_n)(x) \geq - \int_s^r (G_{\mathcal{B}}(t, \sigma)(\phi'_n(\psi)\mathcal{A}(\sigma)\psi))(x)d\sigma, \quad (4.13)$$

for any $(t, r), (t, s) \in \Lambda \cap J^2$, $x \in \Omega$ and any $n \in \mathbb{N}$. In the rest of this step s, r, t , x and n are arbitrarily fixed as above.

Let $G_k(t, s)$ be the evolution operator associated in $C_b(\Omega)$ with $(\mathcal{A}^{(k)}, \mathcal{B}^{(k)})$, introduced in the proof Proposition 3.3. Applying Theorem A.1, with f being replaced with $\zeta_n := \psi_n - n - 1$, observing that $\zeta_n - G_k(t, s)\zeta_n \leq \psi_n - G_k(t, s)\psi_n$ and recalling that $G_k(t, \sigma)$ preserves positivity, we deduce that

$$\begin{aligned} \psi_n(x) - (G_k(t, s)\psi_n)(x) &\geq - \int_s^t (G_k(t, \sigma)((\mathcal{A}^{(k)}(\sigma) + c^{(k)}(\sigma, \cdot))\psi_n)(x)d\sigma \\ &\quad + \int_s^t (G_k(t, \sigma)(c^{(k)}(\sigma, \cdot)\psi_n\vartheta_m))(x)d\sigma, \end{aligned} \quad (4.14)$$

for any $k, m \in \mathbb{N}$ and $I \in s < t$, where $\vartheta_m \in C_c(\Omega)$ is supported in Ω_m and its image is contained in $[0, 1]$. Here, we have taken into account that $c^{(k)}(\sigma, \cdot)\psi_n \geq c^{(k)}(\sigma, \cdot)\psi_n\vartheta_m$ and that $G_k(t, \sigma)$ preserves positivity. Writing (4.14) with $t = r$ and applying Lemma A.4, with $T = G_k(t, r)$ (noting that the functions $G_k(r, \cdot)((\mathcal{A}^{(k)} + c^{(k)}(\sigma, \cdot)\psi_n)$ and $G_k(r, \cdot)(c^{(k)}(\sigma, \cdot)\psi_n\vartheta_m)$ are continuous in $\Omega_{I \cap (-\infty, r]}$, yields

$$\begin{aligned} (G_k(t, r)(\psi_n))(x) - (G_k(t, s)\psi_n)(x) &\geq - \int_s^r (G_k(t, \sigma)((\mathcal{A}^{(k)}(\sigma) + c^{(k)}(\sigma, \cdot))\psi_n)(x)d\sigma \\ &\quad + \int_s^r (G_k(t, \sigma)(c^{(k)}(\sigma, \cdot)\psi_n\vartheta_m))(x)d\sigma. \end{aligned} \quad (4.15)$$

We now want to let $k \rightarrow +\infty$ in the first and last side of (4.14) and (4.15). For this purpose, we observe that, for k large enough (which is independent of $\sigma \in [s, r]$) $(\mathcal{A}^{(k)}(\sigma) + c^{(k)}(\sigma, \cdot))\psi_n = (\mathcal{A}(\sigma) + c(\sigma))\psi_n \in C_b(\Omega)$ and $c^{(k)}\psi_n\vartheta_m = c\psi_n\vartheta_m \in C_b(\Omega)$. Therefore, taking Remark 4.4 into account, by dominated convergence we conclude that formula (4.15) holds true with $G_k(\cdot, \cdot)$, $\mathcal{A}^{(k)}$ and $c^{(k)}$ being replaced, respectively, by $G_{\mathcal{B}}(\cdot, \cdot)$, \mathcal{A} and c . Letting $m \rightarrow +\infty$ by monotone convergence, shows that $(G_{\mathcal{B}}(t, \sigma)(c(\sigma, \cdot)\psi_n))(x)$ is finite for almost every $\sigma \in (s, t)$, $(G_{\mathcal{B}}(t, \cdot)(c\psi_n\vartheta_m))(x)$ tends to $(G_{\mathcal{B}}(t, \cdot)(c\psi_n))(x)$ in $L^1((s, t))$ as $m \rightarrow +\infty$ and

$$(G_{\mathcal{B}}(t, r)\psi_n)(x) - (G_{\mathcal{B}}(t, s)\psi_n)(x) \geq - \int_s^r (G_{\mathcal{B}}(t, \sigma)(\mathcal{A}(\sigma)\psi_n))(x)d\sigma.$$

Using the inequality $r\psi'_n(r) - \psi_n(r) \leq 0$ for any $r \geq 0$ (which easily follows recalling that ϕ_n is concave) we deduce that $\mathcal{A}\psi_n \leq \phi'_n(\psi)\mathcal{A}\psi$ and, hence, $G_{\mathcal{B}}(t, \cdot)\mathcal{A}\psi_n \leq G_{\mathcal{B}}(t, \cdot)(\phi'_n(\psi)\mathcal{A}\psi)$, and this latter function is integrable in (s, r) since it differs from $G_{\mathcal{B}}(t, \cdot)\mathcal{A}\psi_n$ in bounded terms. Estimate (4.13) now follows.

Step 2. Here, using the results in Step 1, we prove that $(G_{\mathcal{B}}(t, \sigma)\mathcal{A}(\sigma)\psi)(x)$ is well defined for any $\sigma \in J \cap (-\infty, t]$ and

$$(G_{\mathcal{B}}(t, r)\psi)(x) - (G_{\mathcal{B}}(t, s)\psi)(x) \geq - \int_s^r (G_{\mathcal{B}}(t, \sigma)\mathcal{A}(\sigma)\psi)(x)d\sigma. \quad (4.16)$$

We begin by observing that, by monotone convergence, $(G_{\mathcal{B}}(t_2, t_1)\psi_n)(x)$ tends to $(G_{\mathcal{B}}(t_2, t_1)\psi)(x)$ as $n \rightarrow +\infty$, for any $(t_2, t_1) \in \Lambda$ and any $x \in \Omega$. This limit might

be, *a priori*, $+\infty$. We will show that this is not the case. For this purpose, we use (4.13) with $r = t$, to infer that

$$\begin{aligned} \psi_n(x) &\geq - \int_s^t (G_{\mathcal{B}}(t, \sigma)(\chi_{\{\mathcal{A}(\sigma)\psi > 0\}} \phi'_n(\psi) \mathcal{A}(\sigma)\psi)(x) d\sigma \\ &\quad + \int_s^t (G_{\mathcal{B}}(t, \sigma)(\chi_{\{\mathcal{A}(\sigma)\psi < 0\}} \phi'_n(\psi) |\mathcal{A}(\sigma)\psi|)(x) d\sigma. \end{aligned} \quad (4.17)$$

Since the set $\{(\sigma, x) \in \Omega_J : (\mathcal{A}(\sigma)\psi)(x) > 0\}$ is bounded (due to our assumption on ψ), $G_{\mathcal{B}}(t, \sigma)(\chi_{\{\mathcal{A}(\sigma)\psi > 0\}} \phi'_n(\psi) \mathcal{A}(\sigma)\psi)(x)$ converges in a dominated way to $G_{\mathcal{B}}(t, \sigma)(\chi_{\{\mathcal{A}(\sigma)\psi > 0\}}(\psi) \mathcal{A}(\sigma)\psi)(x)$ for any $\sigma \in [s, t]$. Estimate (4.17) now show that the sequence $((G_{\mathcal{B}}(t, \sigma)(\chi_{\{\mathcal{A}(\sigma)\psi < 0\}} \phi'_n(\psi) |\mathcal{A}(\sigma)\psi|)(x))$ is bounded in $L^1((s, t))$. Hence, by monotone convergence, it tends to $(G_{\mathcal{B}}(t, \sigma)(\chi_{\{\mathcal{A}(\sigma)\psi < 0\}} |\mathcal{A}(\sigma)\psi|)(x)$ in $L^1((s, t))$. Summing up, we have proved that

$$\lim_{n \rightarrow +\infty} \int_s^t (G_{\mathcal{B}}(t, \sigma)(\phi'_n(\psi) \mathcal{A}(\sigma)\psi)(x) d\sigma = \int_s^t (G_{\mathcal{B}}(t, \sigma)(\mathcal{A}(\sigma)\psi)(x) d\sigma.$$

Using again (4.13) (with $r = t$), we conclude that the sequence $(G_{\mathcal{B}}(t, s)\psi_n)(x)$ is bounded, as claimed.

Now, taking the above results into account, we can let $n \rightarrow +\infty$ in (4.13) and obtain (4.16).

Step 3. Here, we prove that the function $(G_{\mathcal{B}}(t, \cdot)\psi)(x)$ is bounded, in any compact interval contained in the $J \cap (-\infty, t)$, by a constant independent of x . Since $\mathcal{A}\psi \leq c_2 - c_1\psi^{1+\varepsilon}$ in Ω_J it follows that $G_{\mathcal{B}}(t, \cdot)\mathcal{A}\psi \leq c_2 G_{\mathcal{B}}(t, \cdot)\mathbb{1} - c_1 G(t, \cdot)\psi^{1+\varepsilon}$. In particular, this inequality shows that $(G(t, s)\psi^{1+\varepsilon})(x) < +\infty$. Hölder inequality and the fact that $0 < g(t, s, x, \Omega) = (G_{\mathcal{B}}(t, s)\mathbb{1})(x) \leq 1$ for every $t > s$ and $x \in \Omega$ show that $((G(t, s)\psi)(x))^{1+\varepsilon} \leq (G(t, s)\psi^{1+\varepsilon})(x)$ for any $t > s \in I$ and $x \in \Omega$. Hence, from the above results and (4.16), and recalling that $G_{\mathcal{B}}(t, s)\mathbb{1} \leq \mathbb{1}$ for any $(t, s) \in \Lambda$, we get

$$(G_{\mathcal{B}}(t, r)\psi)(x) - (G_{\mathcal{B}}(t, s)\psi)(x) \geq -c_2(r - s) + c_1 \int_s^r ((G_{\mathcal{B}}(t, \sigma)\psi)(x))^{1+\varepsilon} d\sigma. \quad (4.18)$$

Let us set $\zeta(r) := (G_{\mathcal{B}}(t, t - r)\psi)(x)$ for any $r \in [0, \bar{r}]$, where $\bar{r} = t - \inf I$. Estimate (4.18) shows that the function $r \mapsto \zeta(r) - c_2 r$ is decreasing. As a byproduct, ζ admits left and right limits at any point $r \in (0, \bar{r})$. Moreover,

$$\lim_{r \rightarrow r_*^-} \zeta(r) \geq \zeta(r_*) \geq \lim_{r \rightarrow r_*^+} \zeta(r), \quad r_* \in (0, \bar{r}). \quad (4.19)$$

For any $x \in \Omega$, let $y(\cdot; x)$ denote the solution of the differential equation $y'(r) = -c_1(y(r))^{1+\varepsilon} + c_2$, $r > 0$, which satisfies the condition $y(0) = \psi(x)$. Clearly, $y(\cdot; x)$ is defined in $[0, +\infty)$ and

$$\mathcal{H}(y(t)) := \int_{y(t; x)}^{+\infty} \frac{1}{c_1 z^{1+\varepsilon} - c_2} dz \geq \delta, \quad t \geq \delta.$$

Hence $y(t; x) \in \mathcal{H}^{-1}((\delta, +\infty))$, which is a bounded set since $\lim_{\sigma \rightarrow +\infty} \mathcal{H}(\sigma) = 0$. Thus, the function $y(\cdot, x)$ is bounded by $M := \mathcal{H}^{-1}((\delta, \infty))$ for any $x \in \Omega$.

To conclude the proof, let us show that $\zeta \leq y(\cdot; x)$ for any $x \in \Omega$. We argue by contradiction: we suppose that there exist $s_0 \in (0, \bar{r})$ and $x \in \Omega$ such that $\zeta(s_0) > y(s_0; x)$, and we show that $\zeta > y(\cdot; x)$ in $[0, s_0]$ (this, of course, leads to a contradiction since $\zeta(0) = y(0; x) = \varphi(x)$). For this purpose, we begin by observing that there exists $\delta_0 > 0$ such that $\zeta > y(\cdot; x)$ in $[s_0 - \delta_0, s_0]$. Indeed, if this were not the case, there would exist a sequence (s_n) converging to s_0 from the left such that $\zeta(s_n) \leq y(s_n)$ for any $n \in \mathbb{N}$. Letting $n \rightarrow +\infty$ and taking (4.19) into account, we would get to a contradiction. Suppose that $\delta_0 < s_0$. Then, there exists some

$\bar{s} \in [0, s_0)$ such that $y(\bar{s}; x) \geq \zeta(\bar{s})$ and $y(\cdot; x) < \zeta$ in (\bar{s}, s_0) . As a consequence, $\int_{\bar{s}}^s |\zeta(\sigma)|^{1+\varepsilon} d\sigma > \int_{\bar{s}}^s |y(\sigma; x)|^{1+\varepsilon} d\sigma$ for any $s \in (\bar{s}, s_0)$, and, using estimate (4.18) with $r_1 = \bar{s}$, $r_2 = s \in (\bar{s}, s_0)$, we get

$$y(s; x) - \zeta(s) - (y(\bar{s}; x) - \zeta(\bar{s})) \geq c_1 \left(\int_{\bar{s}}^s |\zeta(r)|^{1+\varepsilon} dr - \int_{\bar{s}}^s |y(r; x)|^{1+\varepsilon} dr \right) > 0,$$

which, in its turn, imply that $\zeta < y(\cdot; x)$ in (\bar{s}, s_0) : a contradiction. \square

5. GRADIENT ESTIMATES

This section is devoted to establish some uniform gradient estimates for the function $G_{\mathcal{B}}(t, s)f$. More precisely, our aim consists in proving that, for any $T > s \in I$, there exists a positive constant $C_{s,T}$ such that

$$\|\nabla_x G_{\mathcal{B}}(t, s)f\|_{\infty} \leq \frac{C_{s,T}}{\sqrt{t-s}} \|f\|_{\infty}, \quad t \in (s, T), \quad (5.1)$$

for any $f \in C_b(\Omega)$. In the particular case when $C_{s,T} \leq C(s)$ for some function C bounded from above in any right-halfline $J \subset I$, estimate (5.1) allows us to conclude that, for any $\varepsilon > 0$, there exists $C'_{s,\varepsilon} > 0$ such that

$$\|\nabla_x G_{\mathcal{B}}(t, s)f\|_{\infty} \leq \frac{C'_{s,\varepsilon}}{\sqrt{t-s}} e^{-(c_0-\varepsilon)(t-s)} \|f\|_{\infty}, \quad t \in (s, +\infty), \quad (5.2)$$

for the same f 's as above. Indeed, in this case,

$$\|\nabla_x G_{\mathcal{B}}(t, r)f\|_{\infty} \leq \frac{C(r)}{\sqrt{t-r}} \|f\|_{\infty} \leq \frac{C_s}{\sqrt{t-r}} \|f\|_{\infty}, \quad s \leq r < t \leq r+1, \quad (5.3)$$

for any $f \in C_b(\Omega)$, where $C_s = \sup_{r>s} C(r)$. Now, if $t > s+1$, we split $G_{\mathcal{B}}(t, s)f = G_{\mathcal{B}}(t, t-1)G_{\mathcal{B}}(t-1, s)f$ and use (5.1) to estimate

$$\|\nabla_x G_{\mathcal{B}}(t, s)f\|_{\infty} \leq C_s \|G_{\mathcal{B}}(t-1, s)f\|_{\infty} \leq \tilde{C}_s e^{-c_0(t-s)} \|f\|_{\infty},$$

which, combined with (5.3), yields to (5.2).

Throughout this section, besides Hypotheses 2.1, 2.3 and 2.4 we will consider the following conditions on the domain Ω and the coefficients of the operators \mathcal{A} and \mathcal{B} . In particular, we assume that the boundary operator is independent of t .

Hypotheses 5.1. (i) $\partial\Omega$ is uniformly of class $C^{3+\alpha}$;

(ii) $q_{ij}, b_j, c \in C_{\text{loc}}^{\alpha/2, 1+\alpha}(\Omega_I)$ for some $\alpha \in (0, 1)$ and any $i, j = 1, \dots, d$;

(iii) there exist locally bounded from above functions $L_j, M_1 : I \rightarrow \mathbb{R}$ ($j = 1, \dots, 4$), with L_1, L_2 nonnegative and $L_4 < 1/2$ in I , such that

$$(a) \quad |\nabla_x q_{ij}(t, x)| \leq M_1(t)\eta(t, x), \quad (b) \quad |\nabla_x c(t, x)| \leq L_1(t) + L_2(t)c(t, x), \quad (5.4)$$

and

$$\langle J_x b(t, x)\xi, \xi \rangle \leq (L_3(t) + L_4(t)c(t, x))|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad (5.5)$$

for any $(t, x) \in \Omega_I$;

(iv) for any bounded interval $J \subset I$, there exists $\delta_0 = \delta_0(J)$ such that q_{ij}, b_j and c belong to $C_b^{0,\alpha}(J \times \Omega_{\delta_0})$;

(v) either $(\beta, \gamma) \equiv (0, 1)$ or $\beta \in C_{\text{loc}}^{2+\alpha}(\partial\Omega; \mathbb{R}^d)$ is bounded together with its derivatives, satisfies $\inf_{x \in \partial\Omega} \langle \beta(x), \nu(x) \rangle > 0$ and $\gamma \in C_b^{1+\alpha}(\partial\Omega)$.

Remark 5.2. Note that it is enough to prove estimate (5.1) for functions $f \in C_c^3(\Omega)$. Indeed, if $f \in C_b(\Omega)$ we can find a sequence $(f_n) \in C_c^3(\Omega)$ converging to f locally uniformly in Ω and such that $\|f_n\|_{\infty} \leq \|f\|_{\infty}$ for any $n \in \mathbb{N}$. By Proposition

4.1(i), $\nabla_x G_{\mathcal{B}}(\cdot, s)f_n$ converges pointwise to $\nabla_x G_{\mathcal{B}}(\cdot, s)f$ in $\Omega_{(s,T)}$. Hence, from (5.1), with f being replaced by f_n , we get

$$|(\nabla_x G_{\mathcal{B}}(t, s)f_n)(x)| \leq \frac{C_{s,T}}{\sqrt{t-s}} \|f_n\|_{\infty} \leq \frac{C_{s,T}}{\sqrt{t-s}} \|f\|_{\infty}.$$

Letting $n \rightarrow +\infty$, we obtain (5.1) for $f \in C_b(\Omega)$.

In view of this remark, we will prove (5.1) for functions $f \in C_c^{3+\alpha}(\Omega)$.

Theorem 5.3. *Under Hypotheses 5.1, estimate (5.1) holds true, with the constant $C_{s,T}$ depending on d , η_0 , $\|q_{ij}\|_{C_b^{0,\alpha}((s,T) \times \Omega_{\delta_0})}$, $\|b_j\|_{C_b^{0,\alpha}((s,T) \times \Omega_{\delta_0})}$ ($i, j = 1, \dots, d$), $\|c\|_{C_b^{0,\alpha}((s,T) \times \Omega_{\delta_0})}$, $\sup_{(s,T)} L_j$ ($j = 1, 2, 3, 4$) and $\sup_{(s,T)} M_1$. If all the functions L_j ($j = 1, 2, 3$) and M_1 are bounded from above in $(s, +\infty)$ and $\sup_{(s,+\infty)} L_4 < \frac{1}{2}$, then estimate (5.2) holds true for any $\varepsilon > 0$, and the constant therein appearing is independent of s if in addition the functions M_1 , L_j ($j = 1, 2, 3$) are bounded from above in I , $\sup_I L_4 < \frac{1}{2}$ and q_{ij} , b_j ($i, j = 1, \dots, d$) belong to $C_b^{0,\alpha}(I \times \Omega_{\delta_0})$.*

Proof. Fix $T > s \in I$ and $f \in C_c^3(\mathbb{R}^d)$. We split the proof into two steps. In the first one, we prove a uniform gradient estimate for $G_{\mathcal{B}}(t, s)f$ near the boundary of Ω . More precisely, we prove estimate (5.1) with Ω being replaced by Ω_{δ_1} for a suitable $\delta_1 > 0$. Here, the smoothness of the domain suggests to go back, by means of local charts (and Lemma A.2), to smooth bounded domains of \mathbb{R}_+^d and to consider problems therein defined. In the second step, we prove an interior gradient estimate, i.e., we show that estimate (5.1) is satisfied with Ω being replaced by $\Omega \setminus \Omega_{\delta_1}$. Clearly, combining the results in Steps 1 and 2, (5.1) follows at once.

Throughout the proof, we denote by C positive constants, which are independent of n and $x_0 \in \partial\Omega$, which may vary from line to line.

Step 1. We first consider the case when \mathcal{B} is a first-order boundary operator. We fix $0 < \delta_1 < \min\{\delta_0, r_0\}$, where r_0 is as in Lemma A.2 and $\delta_0 = \delta_0((s, T))$ is given by Hypothesis 5.1(iv), and prove estimate (5.1), with Ω being replaced by Ω_{δ_1} . Clearly, since $\bigcup_{x_0 \in \partial\Omega} B_{\delta_1}(x_0) = \Omega_{\delta_1}$, it suffices to prove that there exists a positive constant $K_{s,T}$, independent of x_0 , such that

$$\|\nabla_x G_{\mathcal{B}}(t, s)f\|_{L^\infty(\Omega \cap B_{\delta_1}(x_0))} \leq \frac{K_{s,T}}{\sqrt{t-s}} \|f\|_{\infty}, \quad t \in (s, T), \quad x_0 \in \partial\Omega. \quad (5.6)$$

Fix $x_0 \in \partial\Omega$, $r_1 \in (\delta_1, r_0)$ and define $R_n = 2\delta_1 - r_1 + (r_1 - \delta_1) \sum_{k=0}^n 2^{-k}$ for any $n \in \mathbb{N} \cup \{0\}$. Using Lemma A.3, we determine a sequence $(\vartheta_n) \subset C_c^\infty(\mathbb{R}^d)$ such that $\chi_{\phi_{x_0}(B_{R_n}(x_0) \cap \Omega)} \leq \vartheta_n \leq \chi_{\phi_{x_0}(B_{R_{n+1}}(x_0) \cap \bar{\Omega})}$ (ϕ_{x_0} is as in Lemma A.2), $D_d \vartheta_n \equiv 0$ on $\partial\mathbb{R}_+^d$ and

$$\|\vartheta_n\|_{C_b^k(\mathbb{R}^d)} \leq \frac{2^{kn}C}{(r_1 - \delta_1)^k}, \quad k = 1, 2, 3. \quad (5.7)$$

Again by Lemma A.3, we fix a smooth function ζ such that $\chi_{\phi_{x_0}(B_{R_{n+1}}(x_0) \cap \Omega)} \leq \zeta \leq \chi_{\phi_{x_0}(B_{r_0}(x_0) \cap \Omega)}$. Since the support of the function $w_n = \vartheta_n v_{x_0} := \vartheta_n(G_{\mathcal{B}}(\cdot, s)f)(\phi_{x_0}^{-1})$ is contained in $\phi_{x_0}(B_{R_{n+1}}(x_0) \cap \Omega)$, a long but straightforward computation reveals that w_n solves the Cauchy problem

$$\begin{cases} D_t w_n(t, x) = (\hat{A} w_n)(t, x) + \hat{g}_n(t, x), & (t, x) \in (s, T) \times \mathbb{R}_+^d, \\ D_d w_n(t, x) + \omega(x) w_n(t, x) = 0, & (t, x) \in (s, T) \times \partial\mathbb{R}_+^d \\ w_n(s, x) = \hat{f}_n(x), & x \in \mathbb{R}_+^d, \end{cases}$$

where $\hat{A} = \text{Tr}(\hat{Q}D^2) + \langle \hat{b}, \nabla_x \rangle - \hat{c}$, with $\hat{Q} = \zeta J\phi_{x_0}(\phi_{x_0}^{-1})Q(\cdot, \phi_{x_0}^{-1})(J\phi_{x_0}(\phi_{x_0}^{-1}))^T + (1 - \zeta)I$, $\hat{b} = \zeta[(J\phi_{x_0}(\phi_{x_0}^{-1})b(\cdot, \phi_{x_0}^{-1}))_h + \text{Tr}(Q(\cdot, \phi_{x_0}^{-1})D^2\phi_{x_0}^h(\phi_{x_0}^{-1}))]$ ($h = 1, \dots, d$), $\hat{c} = \zeta c(\cdot, \phi_{x_0}^{-1})$, $\omega = \zeta \gamma(\phi_{x_0}^{-1})/\rho_{x_0}(\phi_{x_0}^{-1})$ (ρ_{x_0} is defined in (A.10)). Finally, $\hat{g}_n =$

$-2\langle \hat{Q}\nabla\vartheta_n, \nabla_x v_{x_0} \rangle - v_{x_0}(\hat{A} + \hat{c})\vartheta_n$ and $\hat{f}_n = \vartheta_n f(\phi_{x_0}^{-1})$, defined in the whole of \mathbb{R}_+^d . Note that the coefficients of the operator \hat{A} and the function ω are smooth and bounded.

Denote by $G_{\mathcal{R}}(t, s)$ the evolution operator associated to \hat{A} in $C_b(\mathbb{R}_+^d)$ with homogeneous Robin boundary conditions. Using the optimal Schauder estimates $\|G_{\mathcal{R}}(t, s)\psi\|_2 \leq C(t-s)^{-\frac{3}{4}}\|\psi\|_{1/2}$, which holds for any $t \in (s, T]$ and $\psi \in C_b^{1/2}(\mathbb{R}_+^d)$ (where, from now on, we simply write $\|\cdot\|_\beta$ to denote the norm in $C_b^\beta(\mathbb{R}_+^d)$) and the variation-of-constants formula, we can estimate

$$\begin{aligned} & (t-s)\|D_x^2 w_n(t, \cdot)\|_\infty \\ & \leq (t-s) \left[\|D_x^2 G_{\mathcal{R}}(t, s)\hat{f}_n\|_\infty + \left\| \int_s^t (D_x^2 G_{\mathcal{R}}(t, r)\hat{g}_n(r, \cdot))(\cdot) dr \right\|_\infty \right] \\ & \leq C \left\{ \|f\|_\infty + (t-s) \int_s^t (t-r)^{-\frac{3}{4}} \|\hat{g}_n(r, \cdot)\|_{\frac{1}{2}} dr \right\}, \end{aligned} \quad (5.8)$$

for any $t \in (s, T]$. Since $w_{n+1} \equiv v_{x_0}$ in $\phi_{x_0}(B_{R_{n+1}}(x_0) \cap \Omega)$ and $\hat{g}_n(r, \cdot)$ is supported in $\phi(B_{R_{n+1}}(x_0) \cap \Omega)$, for any $r \in (s, T)$ we have

$$\begin{aligned} \|\hat{g}_n(r, \cdot)\|_{\frac{1}{2}} & \leq C \|\vartheta_n\|_{C_b^3(\mathbb{R}^d)} \left(\|\nabla_x w_{n+1}(r, \cdot)\|_{\frac{1}{2}} + \|w_{n+1}(r, \cdot)\|_{\frac{1}{2}} \right) \\ & \leq 8^n C \left((r-s)^{-\frac{3}{4}} \sup_{\sigma \in (s, T)} (\sigma-s)^{\frac{3}{4}} \|\nabla_x w_{n+1}(\sigma, \cdot)\|_{\frac{1}{2}} + \|f\|_\infty \right), \end{aligned}$$

where the constant C depends on $\|q_{ij}\|_{C_b^{0,\alpha}((s,T) \times \Omega_{\delta_0})}$ and $\|b_j\|_{C_b^{0,\alpha}((s,T) \times \Omega_{\delta_0})}$ ($i, j = 1, \dots, d$). Here, we have used the estimate $\|w_{n+1}(r, \cdot)\|_{\frac{1}{2}} \leq 3\|w_{n+1}(r, \cdot)\|_\infty + \|\nabla_x w_{n+1}(r, \cdot)\|_{\frac{1}{2}}$ and (5.7). Thus, it follows that

$$(t-s) \int_s^t (t-r)^{-\frac{3}{4}} \|\hat{g}_n(r, \cdot)\|_{\frac{1}{2}} dr \leq 8^n C \left(\sup_{\sigma \in (s, T)} (\sigma-s)^{\frac{3}{4}} \|\nabla_x w_{n+1}(\sigma, \cdot)\|_{\frac{1}{2}} + \|f\|_\infty \right), \quad (5.9)$$

for any $t \in (s, T)$. Since $\|\nabla_x w_{n+1}(r, \cdot)\|_{\frac{1}{2}} \leq C\|w_{n+1}(r, \cdot)\|_\infty^{\frac{1}{4}}\|D_x^2 w_{n+1}(r, \cdot)\|_\infty^{\frac{3}{4}}$, using estimate (4.1) and Young inequality, we deduce that

$$\sup_{r \in (s, T)} (r-s)^{\frac{3}{4}} \|\nabla_x w_{n+1}(r, \cdot)\|_{\frac{1}{2}} \leq C a_{n+1}^{\frac{3}{4}} \|w_{n+1}\|_\infty^{\frac{1}{4}} \leq \varepsilon a_{n+1} + C\varepsilon^{-3} \|f\|_\infty, \quad (5.10)$$

for any $\varepsilon > 0$ and $n \in \mathbb{N} \cup \{0\}$, where $a_k := \sup_{t \in (s, T)} (t-s)\|D^2 w_k(t, \cdot)\|_\infty$ for any $k \in \mathbb{N} \cup \{0\}$. Now, replacing (5.9) and (5.10) in (5.8) we obtain

$$a_n \leq 8^n C (\varepsilon a_{n+1} + \varepsilon^{-3} \|f\|_\infty), \quad n \in \mathbb{N} \cup \{0\}, \quad \varepsilon > 0. \quad (5.11)$$

The classical Schauder estimates in [11, Thm. IV.10.1] and (4.1) show that $\|v_{x_0}(r, \cdot)\|_{C^2(\phi(B_{r_1}(x_0) \cap \Omega))} \leq C\|f\|_\infty$, for any $r \in (s, T)$ where C depends also on $\|c\|_{C_b^{0,\alpha}((s,T) \times \Omega_{\delta_0})}$. It thus follows that $a_n \leq 4^n C\|f\|_\infty$ for any $n \in \mathbb{N} \cup \{0\}$. We can now choose $\varepsilon > 0$ in (5.11) such that $\tau := \varepsilon 8^n C < 2^{-9}$. Multiplying both the sides of (5.11) by τ^n and summing over $n \in \mathbb{N}$, we realize that the two series converge (in view of the above estimate on a_n) and, as a by product, we deduce that

$$(t-s)\|D_x^2 w_0(t, \cdot)\|_\infty \leq C\|f\|_\infty, \quad t \in (s, T). \quad (5.12)$$

Since $\|\nabla_x w_0(t, \cdot)\|_\infty \leq C\|w_0(t, \cdot)\|_\infty^{1/2}\|D_x^2 w_0(t, \cdot)\|_\infty^{1/2}$, from (5.12) it follows immediately that $\sqrt{t-s}\|\nabla_x w_0(r, \cdot)\|_\infty \leq C\|f\|_\infty$ for any $t \in (s, T)$. Recalling that $\vartheta_0 \equiv 1$ in $\phi(B_{\delta_1}(x_0) \cap \Omega)$ we conclude that $\sup_{t \in (s, T)} \sqrt{t-s}\|\nabla_x v_{x_0}(t, \cdot)\|_{\phi(B_{\delta_1}(x_0) \cap \Omega)} \leq C\|f\|_\infty$. Now, taking into account that $\nabla_x v_{x_0} = (J\phi_{x_0}^{-1})^T(\nabla_x G_{\mathcal{B}}(\cdot, s)f)(\phi_{x_0}^{-1})$, estimate (5.6) follows at once.

In the case of homogeneous Dirichlet boundary conditions, the proof is completely similar. Actually, Lemma A.2 is not needed here, since one can use the covering $\{\psi_h : h \in \mathbb{N}\}$ of $\partial\Omega$.

Step 2. We fix two functions $\vartheta_1, \vartheta_2 \in C^\infty(\mathbb{R}^d)$ such that $\chi_{\Omega \setminus \Omega_{\delta_1}} \leq \vartheta_1 \leq \chi_{\Omega \setminus \Omega_{\delta_1/2}}$ and $\chi_{\Omega \setminus \Omega_{\delta_1/2}} \leq \vartheta_2 \leq \chi_{\Omega \setminus \Omega_{\delta_1/4}}$. We denote by v the trivial extension to the whole of \mathbb{R}^d of the function $\vartheta_1 G_{\mathcal{B}}(\cdot, s)f$. As it is easily seen, the function v solves the Cauchy problem

$$\begin{cases} D_t v(t, x) = (\tilde{A}v)(t, x) + \psi(t, x), & t \in (s, +\infty), x \in \mathbb{R}^d, \\ v(s, x) = \tilde{f}(x), & x \in \mathbb{R}^d, \end{cases}$$

where ψ (resp. \tilde{f}) is the trivial extension to the whole of $(s, +\infty) \times \mathbb{R}^d$ (resp. \mathbb{R}^d) of the function $\psi = -(G_{\mathcal{B}}(\cdot, s)f)(\mathcal{A} + c)\vartheta_1 - 2\langle Q\nabla_x G_{\mathcal{B}}(\cdot, s)f, \nabla \vartheta_1 \rangle$ (resp. $\vartheta_1 f$) and $\tilde{A}(t) = \text{Tr}(\tilde{Q}(t, \cdot)D^2) + \langle \tilde{b}(t, \cdot), \nabla_x \rangle + \tilde{c}(t, \cdot)$, where $\tilde{Q} = \vartheta_2 Q + (1 - \vartheta_2)I$, $\tilde{b} = \vartheta_2 b$ and $\tilde{c} = \vartheta_2 c$. Since the continuous function ψ is supported in Ω_{δ_1} , in view of the boundedness assumptions on the diffusion and drift coefficients, the definition of the function ϑ_1 and Step 1,

$$\|\psi(t, \cdot)\|_\infty \leq \frac{C}{\sqrt{t-s}} \|f\|_\infty, \quad t \in (s, T). \quad (5.13)$$

Therefore, arguing as in Step 1, we can easily show that

$$\nabla_x v(t, x) = (\nabla_x G(t, s)\tilde{f})(x) + \int_s^t (\nabla_x G(t, r)\psi(r, \cdot))(x)dr, \quad (5.14)$$

for any $(t, x) \in (s, T) \times \mathbb{R}^d$, where $G(t, s)$ denotes the evolution operator associated to the operator \tilde{A} in $C_b(\mathbb{R}^d)$ (see [2]).

We claim that there exists a positive constant C , independent of f , such that

$$|(\nabla_x G(t, s)g)(x)| \leq \frac{C}{\sqrt{t-s}} \|g\|_\infty, \quad t \in (s, T), x \in \mathbb{R}^d, \quad (5.15)$$

for any $g \in C_c(\mathbb{R}^d)$. Once this estimate is proved, from (5.13) and (5.14) it follows that $\sqrt{t-s}\|\nabla_x v(t, \cdot)\|_\infty \leq C\|f\|_\infty$ for any $t > s$, from which the gradient estimate for $G_{\mathcal{B}}(t, s)f$ in $\Omega \setminus \Omega_{\delta_1}$ follows immediately, recalling that $v \equiv G_{\mathcal{B}}(\cdot, s)f$ in $(s, +\infty) \times (\Omega \setminus \Omega_{\delta_1})$.

To prove (5.15), we fix $g \in C_c(\mathbb{R}^d)$ and, for any $n \in \mathbb{N}$ such that $\text{supp}(g) \subset B_n$, we introduce the evolution operator $G_n^N(t, s)$ associated to \tilde{A} , with homogeneous Neumann boundary conditions, in $C_b(B_n)$. By [2, Thm. 2.3] $\nabla_x G_n^N(\cdot, s)g$ converges to $\nabla_x G(\cdot, s)g$ pointwise in $(s, T] \times \mathbb{R}^d$. Let $z_n \in C_b([s, T] \times \overline{B_n}) \cap C^{1,2}((s, T) \times B_n)$ be the function defined by $z_n(t, x) := (u(t, x))^2 + a(t-s)|(\nabla_x u(t, x))^2|$ for any $(t, x) \in [s, T] \times B_n$, where $u := G_n^N(\cdot, s)g$ and the constant a will be chosen later on. Since the matrix $J\nu$ is positive definite, the normal derivative of z_n is nonpositive on ∂B_n (see the proof of Theorem 5.4 for further details). A simple computation shows that z_n satisfies problem

$$\begin{cases} D_t z_n(t, x) = (\tilde{A}z_n)(t, x) + \psi_n(t, x), & t \in (s, T], x \in B_n \\ \frac{\partial z_n}{\partial \nu}(t, x) \leq 0, & t \in (s, T], x \in \partial B_n, \\ z_n(s, x) = (g(x))^2, & x \in B_n, \end{cases}$$

where

$$\psi_n = a|\nabla_x u|^2 - \tilde{c}u^2 - 2\langle \tilde{Q}\nabla_x u, \nabla_x u \rangle - a(\cdot - s)\tilde{c}|\nabla_x u|^2 - 2a(\cdot - s)\text{Tr}(D_x^2 u \tilde{Q} D_x^2 u)$$

$$+ 2a(\cdot - s) \sum_{i,j,k=1}^d D_k \tilde{q}_{ij} D_k u D_{ij} u + 2a(\cdot - s) [\langle J_x \tilde{b} \nabla_x u, \nabla_x u \rangle - u \langle \nabla_x \tilde{c}, \nabla_x u \rangle]. \quad (5.16)$$

Notice that the coefficients of the operator $\tilde{A}(t)$ satisfy Hypothesis 5.1(iii) with the same values of L_2 and L_4 and with M_1 , η , L_1 and L_3 being replaced, respectively, by $\tilde{M}_1 = M_1 \chi_\Omega + \max\{1, \eta_0^{-1}\} \|\nabla \vartheta_2\|_\infty (\|q_{ij}\|_{C_b((s,T) \times \Omega_{\delta_0})} + 1) \chi_{\Omega_{\delta_1}}$, $\tilde{\eta} = \vartheta_2 \eta + 1 - \vartheta_2$, $\tilde{L}_1 = \vartheta_2 L_1 + \|\nabla \vartheta_2\|_\infty \|c\|_{C_b((s,T) \times \Omega_{\delta_0})}$ and $\tilde{L}_3 = \vartheta_2 L_3 + \|\nabla \vartheta_2\|_\infty \|b\|_{C_b((s,T) \times \Omega_{\delta_0})}$. Therefore, we can estimate

$$\langle \tilde{Q} \nabla_x u, \nabla_x u \rangle \geq \tilde{\eta} |\nabla_x u|^2, \quad \text{Tr}(D_x^2 u(t, \cdot) \tilde{Q}(t, \cdot) D_x^2 u(t, \cdot)) \geq \tilde{\eta} |D_x^2 u|^2, \quad (5.17)$$

$$\left| \sum_{i,j,k=1}^d D_k \tilde{q}_{ij} D_k u D_{ij} u \right| \leq \tilde{M}_1 d \tilde{\eta} |\nabla u| |D_x^2 u|, \quad |\langle \nabla_x \tilde{c}, \nabla_x u \rangle| \leq (\tilde{L}_1 + L_2 \tilde{c}) |\nabla_x u|, \quad (5.18)$$

$$\langle J_x \tilde{b} \nabla_x u, \nabla_x u \rangle \leq (\tilde{L}_3 + L_4 \tilde{c}) |\nabla_x u|^2, \quad (5.19)$$

in $\Omega_{(s,T)}$. Now, estimating

$$|\nabla_x u| |D_x^2 u| \leq \varepsilon |D_x^2 u|^2 + \frac{1}{4\varepsilon} |\nabla_x u|^2, \quad |u| |\nabla_x u| \leq \varepsilon |\nabla_x u|^2 + \frac{1}{4\varepsilon} u^2,$$

for any $\varepsilon > 0$, from (5.16)-(5.19) we deduce that

$$\begin{aligned} \psi_n &\leq \frac{a}{2\varepsilon} (T-s) \bar{L}_1 u^2 + \left(\frac{a}{2\varepsilon} (T-s) \bar{L}_2 - 1 \right) \tilde{c} u^2 \\ &\quad + \left[a + \left(a(T-s) \frac{\bar{M}_1 d}{2\varepsilon} - 2 \right) \tilde{\eta} + 2a(T-s) (\bar{L}_3 + \bar{L}_1 \varepsilon) \right] |\nabla_x u|^2 \\ &\quad + a(T-s) (2\varepsilon \bar{L}_2 + 2\bar{L}_4 - 1)^+ \tilde{c} |\nabla_x u|^2 + 2a(T-s) (\bar{M}_1 d \varepsilon - 1)^+ \tilde{\eta} |D_x^2 u|^2, \end{aligned}$$

for any $\varepsilon > 0$, where $\bar{L}_{2j-1} = \sup_{(s,T)} \tilde{L}_{2j-1}$, $\bar{L}_{2j} = \sup_{(s,T)} L_{2j}$ ($j = 1, 2$), $\bar{M}_1 = \sup_{(s,T)} \tilde{M}_1$. Thus, choosing $\varepsilon = \frac{1}{2} \min \left\{ \frac{1-2\bar{L}_4}{\bar{L}_2}, \frac{1}{\bar{M}_1 d} \right\}$, we can make nonpositive the coefficients in front of both $\tilde{c} |\nabla_x u|^2$ and $|D_x^2 u|^2$. Observing that the coefficients in front of $\tilde{c} u^2$ and $|\nabla_x u|^2$ tend, respectively, to -1 and $-2\tilde{\eta} < -2\tilde{\eta}_0$, as $a \rightarrow 0^+$, we can then choose a small enough such that these coefficients are negative. With these choices of ε and a , we deduce that $\psi_n \leq H_{s,T} u^2 \leq H_{s,T} z_n$ for any $n \in \mathbb{N}$ and some positive constant $H_{s,T}$, depending on \bar{L}_j ($j = 1, 2, 3, 4$), \bar{M}_j ($j = 1, 2$), η_0 , d , s and T . By applying the classical maximum principle to the function $(t, x) \mapsto e^{-H_{s,T}(t-s)} z_n(t, x)$ we conclude that $e^{-H_{s,T}(t-s)} z_n \leq \|g\|_\infty^2$, i.e.,

$$((G_n^N(t, s)g)(x))^2 + a(t-s) |(\nabla_x G_n^N(t, s)g)(x)|^2 \leq C \|g\|_\infty^2, \quad (t, x) \in [s, T] \times B_n.$$

Letting $n \rightarrow +\infty$ we get (5.15). \square

In the following subsection, we consider the particular cases when the operator \mathcal{A} is endowed with Neumann and Robin boundary conditions. In the first case we show that the boundedness assumptions on its coefficient in a neighborhood of $\partial\Omega$ and the additional smoothness condition on Ω can be removed provided that Ω is convex.

5.1. Neumann boundary conditions.

Theorem 5.4. *Let Ω be a convex open set. Then, under Hypotheses 5.1(ii), (iii), estimate (5.1) holds true with the constant $C_{s,T}$ depending also on $\sup_{(s,T)} M_1$, $\sup_{(s,T)} L_j$ ($j = 1, 2, 3, 4$). If the functions L_j ($j = 1, 2, 3$) and M_1 are bounded from above in $(s, +\infty)$ and $\sup_{(s, +\infty)} L_4 < \frac{1}{2}$, then estimate (5.2) holds true for*

any $\varepsilon > 0$, and the constant therein appearing is independent of s if further L_j ($j = 1, 2, 3$), M_1 are bounded from above in I and $\sup_I L_4 < \frac{1}{2}$.

Proof. The proof is an adaption to the nonautonomous case of the gradient estimates in [5].

Fix $T > s \in I$, $f \in C_c^3(\Omega)$ and an increasing sequence (Ω_n) of bounded, smooth convex sets such that $\lim_{n \rightarrow +\infty} \Omega_n = \Omega$ and $\partial\Omega \cap \partial\Omega_n \neq \emptyset$ for any $n \in \mathbb{N}$. Denote by $G_n^N(t, s)$ the evolution operator in $C_b(\Omega_n)$ associated with \mathcal{A} with homogeneous Neumann boundary conditions on $\partial\Omega_n$. Adapting the arguments in the proof of Theorem 3.4, we can easily prove that $G_n^N(\cdot, s)f$ converges to $G_{\mathcal{B}}(\cdot, s)f$ in $C^{1,2}(K)$ for any compact set $K \subset \overline{\Omega_{(s, +\infty)}}$. Since the normal derivative of $G_n^N(t, s)f$ identically vanishes on $\partial\Omega_n$, each tangential derivative on $\partial\Omega$ of $\frac{\partial}{\partial\nu}G_n^N(t, s)f$ vanishes. Therefore, $\langle (D^2G_n^N(t, s)f)(x)\nu(x), \tau \rangle + \langle J\nu(x)\tau, (\nabla_x G_n^N(t, s)f)(x) \rangle = 0$ for any vector τ tangent to $\partial\Omega$ at x , and any $x \in \partial\Omega$. In particular, taking $\tau = (\nabla_x G_n^N(t, s)f)(x)$ and recalling that, since Ω_n is convex, the quadratic form associated with the matrix $J\nu$ is everywhere nonnegative on $\partial\Omega_n$, we conclude that $\langle (D^2G_n^N(t, s)f)(x)(\nabla_x G_n^N(t, s)f)(x), \nu(x) \rangle \leq 0$ for any $x \in \partial\Omega_n$. Therefore, the function $|\nabla_x G_n^N(t, s)f|^2$ has nonpositive normal derivative on $\partial\Omega$. As a byproduct, for any $n \in \mathbb{N}$ the function $z_n = |G_n^N(\cdot, s)f|^2 + a(\cdot - s)|\nabla_x G_n^N(\cdot, s)f|^2$ has a nonpositive normal derivative on $\partial\Omega_n$. We can now argue as in Step 2 of the proof of Theorem 5.3 and show that, for a suitable choice of the parameter a , the function $D_t z_n - \mathcal{A}z_n$ is nonpositive in $(s, T) \times \Omega_n$. Hence, using the classical maximum principle and letting $n \rightarrow +\infty$, we obtain estimate (5.1). \square

As a consequence of Theorem 5.4 we can prove gradient estimates for solutions to problem $(P_{\mathcal{B}})$ in \mathbb{R}_+^d when \mathcal{A} is endowed with Robin boundary conditions, i.e., when $\mathcal{B} = \mathcal{R} = \frac{\partial}{\partial\nu} + \gamma I$. Besides Hypotheses (5.1)(ii), (iii) we assume the following conditions:

- Hypotheses 5.5.** (i) the diffusion coefficients q_{ij} belongs to $C_b^{0,1}(J \times \mathbb{R}_{+, \delta}^d)$ for some $\delta > 0$ and any bounded interval $J \subset I$;
(ii) there exists a locally bounded function $L_5 : I \rightarrow (0, +\infty)$ such that $|b| \leq L_5(1 + c)$ in $I \times \mathbb{R}_{+, \delta}^d$;
(iii) $\gamma \in C_{\text{loc}}^{2+\alpha}(\overline{\mathbb{R}_+^d})$ and there exist a constant L_6 , a locally bounded from above function $L_7 : I \rightarrow (0, +\infty)$ and a function $\Gamma \in C_{\text{loc}}^{3+\alpha}(\overline{\mathbb{R}_+^d})$ such that Γ is supported in $\mathbb{R}_{+, \delta}^d$, $D_d \Gamma \equiv \gamma$ on $\partial\mathbb{R}_+^d$,

$$\|\nabla \Gamma\|_{\infty} + \|D^2 \Gamma\|_{\infty} + \|D^3 \Gamma\|_{\infty} \leq L_6, \quad (5.20)$$

$$\inf_{\mathbb{R}_{+, \delta}^d} [(\mathcal{A} + c)\Gamma - \langle Q \nabla \Gamma, \nabla \Gamma \rangle] \geq -L_7, \quad (5.21)$$

in I , where δ is as in (i).

Remark 5.6. Sufficient conditions for Hypothesis 5.5(iii) hold are the following:

- (i) the support of γ is contained in a compact set $K \subset \mathbb{R}^{d-1}$. In this case we can take $\Gamma(x) = \gamma(x')\vartheta(x_d)$ for any $x \in \mathbb{R}_+^d$, where ϑ is a smooth nonnegative function supported in $[0, \delta]$ such that $\vartheta \leq 1$ in $[0, \delta]$ and $\vartheta'(0) = 1$;
(ii) $\gamma(x) = \gamma_1(|x'|^2)$ for any $x \in \mathbb{R}_+^d$, where γ_1 is a bounded and not increasing smooth function such that $\gamma_0 := \sup_{t \geq 0} (1 + t)^k (|\gamma_1'(t)| + |\gamma_1''(t)|) < +\infty$ for some $k \in \mathbb{N}$. Further, there exists a positive locally bounded function $L : I \rightarrow \mathbb{R}$ such that $|q_{ij}(t, x)| \leq L(t)(1 + |x'|^2)^{k-1}$, $|q_{id}(t, x)| \leq L(t)(1 + |x'|^2)^{k-1/2}$ ($i, j < d$) and $\langle b'(t, x), x' \rangle \leq L(t)(1 + |x'|^2)^k$ for any $(t, x) \in I \times \mathbb{R}_{+, \delta}^d$ and some $\delta > 0$, where $b = (b', b_d)$. Finally, q_{dd} and b_d are bounded in $\mathbb{R}_{+, \delta}^d$. Indeed, in

this case, with the choice of Γ as in (i), we get

$$\begin{aligned} & ((\mathcal{A} + c)\Gamma) - \langle Q\nabla\Gamma, \nabla\Gamma \rangle \\ & \geq -4d\gamma_0 L - 4(d-1)\gamma_0^2 L - 4\sqrt{d-1}(1 + \|\gamma_1\|_\infty)\gamma_0\|\vartheta'\|_\infty L \\ & \quad - \|b_d\|_\infty\|\gamma_1\|_\infty\|\vartheta'\|_\infty - \|q_{dd}\|_\infty\|\gamma_1\|_\infty(\|\gamma_1\|_\infty\|\vartheta'\|_\infty^2 + \|\vartheta''\|_\infty), \end{aligned}$$

in $I \times \mathbb{R}_{+, \delta}^d$. The local boundedness of the function L yields (5.21).

Theorem 5.7. *Under Hypotheses 5.1(ii), (iii) and Hypotheses 5.5, estimate (5.1) is satisfied and the constant $C_{s,T}$ depends on η_0 , d , $\sup_{(s,T)} L_j$ ($j = 1, \dots, 7$), $\sup_{(s,T)} M_1$, $\max_{1 \leq i, j \leq d} \|q_{ij}\|_{C_b^{0,1}((s,T) \times \mathbb{R}_{+, \delta}^d)}$. Further, if the functions L_j ($j = 1, \dots, 7$) are bounded from above in $(s, +\infty)$ and $q_{ij} \in C_b^{0,1}((s, +\infty) \times \mathbb{R}_{+, \delta}^d)$, estimate (5.2) holds true, and the constant therein appearing is independent of s if L_j ($j = 1, \dots, 7$) are bounded from above in I and $q_{ij} \in C_b^{0,1}(I \times \mathbb{R}_{+, \delta}^d)$.*

Proof. We limit ourselves to proving (5.1) and observe that, for any $f \in C_c^3(\mathbb{R}_+^d)$, the function $v = e^\Gamma G_{\mathcal{R}}(\cdot, s)f$ solves the Cauchy Neumann problem associated with the operator $\tilde{\mathcal{A}}$, defined on smooth functions ζ by $\tilde{\mathcal{A}}(t)\zeta = \text{Tr}(QD^2\zeta) + \langle \tilde{b}, \nabla_x \zeta \rangle - \tilde{c}\psi$, where $\tilde{b} = b - 2Q\nabla\Gamma$ and $\tilde{c} = c + (\mathcal{A} + c)\Gamma - \langle Q\nabla\Gamma, \nabla\Gamma \rangle$. Clearly, $G_{\mathcal{R}}(t, s)f$ satisfies the gradient estimate (5.1) if and only if the function v does. Therefore, in view of Theorem 5.4, we can limit ourselves to checking that the pair $(\tilde{\mathcal{A}}, D_d)$ satisfies Hypotheses 2.1(ii)-(iv), 2.4 and 5.1(ii), (iii). Hypotheses 2.1(ii), 2.1(iv) and Hypothesis 5.1(ii) are clearly satisfied as well as Hypotheses 2.4 with $\tilde{\varphi} = e^\Gamma \varphi$. Next, we note that due to (5.21), there exists a positive constant $M = M_{(s,T)}$ such that $\inf_{I \times \mathbb{R}_+^d} \tilde{c} \geq M$. Without loss of generality, we can assume that $M_{(s,T)} \geq 0$. Indeed, as Remark 2.5 shows, we can always reduce to this situation, replacing v by the function $w = e^{M(\cdot-s)}v$. Finally, let us check Hypotheses 5.1(iii). Estimate (5.4)(a) is obvious. As far (5.4)(b) is concerned, from Hypothesis 5.5(ii), recalling that the support of Γ is contained in $\mathbb{R}^{d-1} \times [0, \delta]$ and observing that $|J_x b| \leq L_3 + L_4 c$, due to (5.5), and $c \leq \tilde{c} + L_7$, we get

$$\begin{aligned} |\nabla_x \tilde{c}| & \leq |\nabla_x c| + d \max_{1 \leq i, j \leq d} \|\nabla_x q_{ij}\|_{C_b((s,T) \times \mathbb{R}_{+, \delta}^d)} (\|\nabla\Gamma\|_\infty^2 + \|D^2\Gamma\|_\infty) \\ & \quad + d \max_{1 \leq i, j \leq d} \|q_{ij}\|_{C_b((s,T) \times \mathbb{R}_{+, \delta}^d)} (\|D^3\Gamma\|_\infty + 2\|\nabla\Gamma\|_\infty \|D^2\Gamma\|_\infty) \\ & \quad + |J_x b| \|\nabla\Gamma\|_\infty + |b| \|D^2\Gamma\|_\infty \\ & \leq L_1 + L_2 c + L_6(L_8 d + L_3 + L_5 + 2dL_6 L_8) + L_6(L_4 + L_5) c \\ & \leq L_1 + L_6(L_8 d + L_3 + L_5 + 2dL_6 L_8) + [L_6(L_4 + L_5) + L_2]^+ L_7 \\ & \quad + [L_6(L_4 + L_5) + L_2]^+ \tilde{c}, \end{aligned}$$

in $I \times \mathbb{R}_+^d$, where $L_8 = \max_{1 \leq i, j \leq d} \|q_{ij}\|_{C_b^{0,1}((s,T) \times \mathbb{R}_{+, \delta}^d)}$. Similarly, taking Hypothesis 5.5(i) and condition (5.20) into account we deduce that

$$\langle J_x \tilde{b} \xi, \xi \rangle \leq (L_3 + 2dL_6 L_8 + L_4^+ L_7 + L_4^+ \tilde{c}) |\xi|^2,$$

in $I \times \mathbb{R}_{+, \delta}^d$ and for any $\xi \in \mathbb{R}^d$. Hence, condition (5.5) is satisfied with L_4^+ replacing L_4 and a different function L_3 . \square

6. EXAMPLES

In this section we provide some class of operators \mathcal{A} which fulfill our assumptions. We confine ourselves to the relevant cases when $\Omega = \mathbb{R}_+^d$ and when Ω is an exterior domains. In what follows I denotes a right halflife (possibly $I = \mathbb{R}$), J any bounded interval contained in I and $\alpha \in (0, 1)$.

Example 6.1. Let \mathcal{A} and \mathcal{B} be, respectively, the elliptic operator, defined by

$$\mathcal{A}(t) = \omega(t)(1 + |x|^2)^r \Delta_x + \langle b(t, x), \nabla_x \rangle - \hat{c}(t, x)(1 + |x|^2)^m, \quad (6.1)$$

for any $(t, x) \in I \times \mathbb{R}_+^d$, and the operator in (1.2). The coefficient ω belongs to $C_{\text{loc}}^{\alpha/2}(I)$, the entries of the vector b and the function \hat{c} belong to $C_{\text{loc}}^{\alpha/2, \alpha}(I \times \mathbb{R}_+^d)$. Moreover, $\inf_I \omega > 0$ and $\inf_{I \times \mathbb{R}_+^d} \hat{c} = c_0 > 0$. Finally, there exist $R > 0$, $p \in [0, +\infty)$, such that $(r - 1)^+ < \max\{p, m\}$, and a function $k_1 : I \rightarrow \mathbb{R}$ with positive infimum over any $J \subset I$, such that $\langle b(t, x), x \rangle \leq -k_1(t)(1 + |x|^2)^p |x|^2$ for any $(t, x) \in I \times (\mathbb{R}_+^d \setminus B_R)$. As far as the coefficients of the operator \mathcal{B} are concerned, we assume that $\beta_i, \gamma \in C_{\text{loc}}^{(1+\alpha)/2, 1+\alpha}(\bar{I} \times \mathbb{R}^{d-1})$ ($i = 1, \dots, d$), $\sum_{i=1}^d \beta_i^2 = 1$, $\gamma \geq 0$ in $I \times \mathbb{R}^{d-1}$ and $\inf_{(t, x') \in I \times \mathbb{R}^{d-1}} \beta_d(t, x') > 0$. Moreover, we assume that $\langle \beta'(t, x'), x' \rangle + \gamma(t, x')(1 + |x'|^2) \geq 0$ for any $(t, x') \in I \times \mathbb{R}^{d-1}$.

Under the previous set of assumptions, the function φ , defined by $\varphi(x) = 1 + |x|^2$ for any $x \in \mathbb{R}_+^d$, satisfies the estimates

$$(\mathcal{A}(t)\varphi)(x) \leq 2d\omega_J(1 + |x|^2)^r - 2k_{1,J}(1 + |x|^2)^p |x|^2 - \hat{c}_0(1 + |x|^2)^{m+1},$$

for any $J \subset I$, $t \in J$ and $x \in \mathbb{R}_+^d \setminus B_R$, where $\omega_J = \sup_J \omega$, $k_{1,J} = \inf_J k_1$. Moreover, $(\mathcal{B}(t)\varphi)(x', 0) = 2\langle \beta'(t, x'), x' \rangle + \gamma(t, x')(1 + |x'|^2)$ for any $t \in J$ and any $x' \in \mathbb{R}^{d-1}$. The assumptions on m, p, r and on β' and γ show that Hypotheses 2.4 hold true. Moreover, the assumptions of Theorem 4.5 are satisfied as well, with $\psi = \varphi$ and $\varepsilon = \max\{p, m\}$. Hence, the operator $\mathcal{G}_{\mathcal{B}}(t, s)$, associated with the operator \mathcal{A} in (6.1), is compact for any $(t, s) \in \Lambda$.

In the particular case when $\mathcal{B} = \frac{\partial}{\partial \nu}$ (i.e., Neumann boundary conditions are prescribed), $b(t, x) = -b_0(t)x(1 + |x|^2)^p$ for some positive function $b_0 \in C_{\text{loc}}^{\alpha/2}(I) \cap C_b(I)$, $\hat{c} \in C_b^{0,1}(I \times \mathbb{R}_+^d)$, the assumptions of Theorem 5.4 are satisfied with $\eta(t, x) = \omega(t)(1 + |x|^2)^r$, $M_1 = r$, $L_1 \equiv L_3 \equiv L_4 \equiv 0$, $L_2 = m + c_0^{-1} \|\nabla_x \hat{c}\|_\infty$. Hence, the gradient estimate (5.1) holds true. If the coefficients ω , b_i ($i = 1, \dots, d$) and \hat{c} belong to $C_b(I)$ and to $C_b^{0,1}(I \times \mathbb{R}_+^d)$, respectively, then the estimate (5.2) holds true as well.

Example 6.2. Let

$$\mathcal{A}(t) = (1 + x_d^2)^r \text{Tr}(\hat{Q}(t, x)D_x^2) + (1 + x_d^2)^p \langle \hat{b}(t, x), \nabla_x \rangle - \hat{c}(t, x)(1 + x_d^2)^m, \quad (6.2)$$

for any $(t, x) \in I \times \mathbb{R}_+^d$, where m, p, r are nonnegative constants such that $r < \max\{p+1, m+1\}$. The function \hat{c} and the entries of \hat{Q} and \hat{b} belong to $C_b^{0,1}(J \times \mathbb{R}_+^d) \cap C_{\text{loc}}^{\alpha/2, 1+\alpha}(I \times \mathbb{R}_+^d)$ for any $J \subset I$ and $0 < c_0(t) := \inf_{\mathbb{R}_+^d} \hat{c}(t, \cdot)$, the function c_0 being locally bounded from below by a positive constant. Moreover, $\langle \hat{Q}(t, x)\xi, \xi \rangle \geq \eta_0|\xi|^2$ for any $(t, x) \in I \times \mathbb{R}_+^d$, $\langle J_x \hat{b}(t, x)\xi, \xi \rangle \leq -\sigma_0(t)|\xi|^2$, $\langle \hat{b}(t, x), x \rangle \leq -\sigma_0(t)|x|^2$ for any $(t, x) \in I \times \mathbb{R}_+^d$, any $J \subset I$, $\xi \in \mathbb{R}^d$ and some continuous function $\sigma_0 : I \rightarrow (0, +\infty)$. Finally, the coefficients of the operator \mathcal{B} in (1.2) satisfy $\beta \in C_{\text{loc}}^{2+\alpha}(\mathbb{R}^{d-1})$, $\gamma \in C_{\text{loc}}^{1+\alpha}(\mathbb{R}^{d-1})$, $\sum_{i=1}^d \beta_i^2 \equiv 1$, $\gamma \geq 0$ in \mathbb{R}^{d-1} , $\inf_{\mathbb{R}^{d-1}} \beta_d > 0$ and $\langle \beta'(x'), x' \rangle + \gamma(x')(k^2 + |x'|^2) \geq 0$ for any $x' \in \mathbb{R}^{d-1}$ and some positive constant $k \geq 1$.

Under this set of assumptions, the function φ , defined by $\varphi(x) = \sqrt{k^2 + |x|^2}$ for any $x \in \mathbb{R}_+^d$, satisfies Hypothesis 2.4. Indeed,

$$\frac{(\mathcal{A}(t)\varphi)(x)}{\varphi(x)} \leq \sqrt{d} \|\hat{Q}\|_{C_b(J \times \mathbb{R}_+^d; \mathbb{R}^{d^2})} (1 + x_d^2)^{r-1} - \frac{\sigma_0(t)|x|^2}{k^2 + |x|^2} (1 + x_d^2)^{p-c_0(t)} (1 + x_d^2)^m,$$

for any $(t, x) \in J \times \mathbb{R}_+^d$ and any $J \subset I$. The choice of m, p, r shows that the right-hand side of the previous inequality tends to $-\infty$ as $x_d \rightarrow +\infty$, uniformly with respect to $t \in J$. Hence, $(\mathcal{A}\varphi)/\varphi$ is bounded in $J \times \mathbb{R}_+^d$. Since, clearly, $\mathcal{B}\varphi \geq 0$ on $\partial\mathbb{R}_+^d$, the function φ satisfies Hypotheses 2.4. Similarly, Hypothesis 5.1(iii) is

satisfied with $\eta(t, x) = \eta_0(1 + x_d^2)^r$, $M_1(t) = \eta_0^{-1}(r\|\hat{q}_{ij}(t, \cdot)\|_\infty + \|\nabla_x \hat{q}_{ij}(t, \cdot)\|_\infty)$, $L_1 \equiv L_4 \equiv 0$, $L_2(t) = m + (c_0(t))^{-1}\|\nabla_x \hat{c}(t, \cdot)\|_\infty$, $L_3(t) = -\sigma_0(t) + 2p\|\hat{b}(t, \cdot)\|_\infty$ if $p \leq 1/2$ and $L_3(t) = \max\{(2p-1)^{2p-1}\|\hat{b}(t, \cdot)\|_\infty^{2p}(\sigma_0(t))^{1-2p}, -\sigma_0(t) + 2p\|\hat{b}(t, \cdot)\|_\infty\}$, otherwise. Hence, the gradient estimate (5.1) holds true. If c_0 and σ_0 are bounded from below in I by a positive constant, $\hat{b}_j, D_i \hat{c} \in C_b(I \times \mathbb{R}_+^d)$ and $q_{ij} \in C_b^{0,1}(I \times \mathbb{R}_+^d)$ ($i, j = 1, \dots, d$), then estimate (5.2) holds true as well.

We now consider the case when $\Omega \subset \mathbb{R}^d$ is an exterior domain.

Example 6.3. Assume that Ω has a boundary uniformly of class $C^{2+\alpha}$. Let \mathcal{A} , \mathcal{B} be the operators in (1.1) and (1.2). Assume that Hypotheses 2.1(ii)-(iv) and Hypotheses 2.3 are satisfied. Assume that

$$\ell_J := \sup_{(t,x) \in J \times (\mathbb{R}_+^d \setminus B_1)} \frac{\text{Tr}(Q(t, x)) + \langle b(t, x), x \rangle}{|x|^2} < +\infty, \quad t \in J, \quad (6.3)$$

for any $J \subset I$. For instance, condition (6.3) is satisfied when Q is bounded in Ω_I and $\langle b(t, x), x \rangle$ grows at infinity at most quadratically, uniformly with respect to $t \in J$ for any bounded interval $J \subset I$.

If $\gamma_0 := \inf_{\Omega_I} \gamma \geq 0$, under the previous assumptions, the function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, defined by $\varphi(x) = (1 - r_\Omega(x))\vartheta(x) + (1 - \vartheta(x))(1 + |x|^2)$ for any $x \in \mathbb{R}^d$, satisfies Hypotheses 2.4. Here, ϑ is any smooth function with $\text{supp}(\vartheta) \subset \Omega_\delta$ and $\vartheta \equiv 1$ in $\Omega_{\delta/2}$, where δ is defined in Remark 2.2(b). If, further, $c(t, x) = \hat{c}(t, x)(1 + |x|^2)^m$ for any $(t, x) \in \Omega_I$, some $m > 1$ and some smooth enough and bounded function \hat{c} with positive infimum on Ω_I , then the function $\psi : \Omega \rightarrow \mathbb{R}$ defined by $\psi(x) = 1 + |x|^2$, $x \in \Omega$, satisfies the assumption of Theorem 4.5 with $\varepsilon = m$. Hence, the evolution operator $G_{\mathcal{B}}(t, s)$ is compact.

Let us now assume that $\gamma_0 < 0$. Fix a function $\zeta \in C_b^{2+\alpha}([0, +\infty))$ with positive infimum, such that $\zeta(0) = 1$, $\zeta \equiv 1/2$ in $[\delta/2, +\infty)$ and $\zeta'(0) < 0$. Further, let σ be a constant greater than $\max\left\{1, \frac{\gamma_0}{\beta_0 \zeta'(0)}\right\}$, where δ is as above. Then, the function Φ , defined by $\Phi(x) = \zeta(\sigma r_\Omega(x))$ for any $x \in \Omega$, belongs to $C_b^{2+\alpha}(\Omega)$ and has positive infimum. Moreover, $(\mathcal{B}(t)\Phi)(x) \geq -\sigma\zeta'(0)\beta_0 + \gamma_0 > 0$ in $I \times \partial\Omega$, due to the choice of σ . Moreover, since Φ is constant outside a neighborhood of $\partial\Omega$, $(\mathcal{A}\Phi)/\Phi$ is bounded in $J \times \Omega$ for any $J \subset I$, i.e., Hypothesis 3.6 is satisfied.

Finally, analogous computations as above show that the function φ , defined by $\varphi(x) = \zeta(\sigma r_\Omega(x)) + (1 - \zeta(\sigma r_\Omega(x))|x|^2$ for any $x \in \Omega$, satisfies Hypotheses 2.4. Therefore, the results in Theorem 3.7 can be applied.

Example 6.4. Assume that $\partial\Omega$ is uniformly of class $C^{3+\alpha}$. Let \mathcal{A} and \mathcal{B} be as in Example 6.2 with x_d^2 being replaced by $|x|^2$. The two functions φ , introduced in Example 6.3, satisfy Hypotheses 2.4. Moreover, arguing as for the operator \mathcal{A} in (6.2), it can be easily shown that Hypotheses 5.1 are satisfied.

APPENDIX A. TECHNICAL RESULTS

Theorem A.1. Let Ω be an unbounded domain with a boundary uniformly of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$. Let \mathcal{A} be the uniformly nonautonomous elliptic operator defined by (1.1), with coefficients in $C^\alpha([a, b], C_b(\overline{\Omega}))$ and let $\mathcal{B} = I$ or $\mathcal{B} = \langle \beta, \nabla \rangle + \gamma$ where $\beta_i, \gamma \in C^\sigma([a, b]; C_b^1(\overline{\Omega}))$ for some $\sigma > 1/2$ and any $(i = 1, \dots, d)$. Then,

$$(G_{\mathcal{B}}(t, s_2)f)(x) - (G_{\mathcal{B}}(t, s_1)f)(x) = - \int_{s_1}^{s_2} (G_{\mathcal{B}}(t, r)\mathcal{A}(r)f)(x) dr, \quad (A.1)$$

for any $s_1, s_2, t \in [a, b]$, with $t \geq \max\{s_1, s_2\}$, any $x \in \Omega$ and any $f \in C_c^2(\Omega)$.

Proof. Estimate (A.1) has been proved in [1, Thms. 2.3(ix) & 6.3] when Ω is bounded, but the arguments used in [1] can be extended to our situation. The two cases being similar, we limit ourselves to dealing with the boundary operator $\langle \beta, \nabla \rangle + \gamma I$.

For any $t \in [a, b]$ we denote by $A(t)$ the realization of $\mathcal{A}(t)$ in $C_b(\overline{\Omega})$ with domain $D(A(t)) = \left\{ u \in \bigcap_{p \geq 1} W_{\text{loc}}^{2,p}(\Omega) \cap C_b(\overline{\Omega}) : \mathcal{A}(t)u \in C_b(\overline{\Omega}), \mathcal{B}(t)u = 0 \text{ on } \partial\Omega \right\}$. To check the assumptions in [1, Thm. 2.3(ix)], we have to prove the following properties:

- (i) there exists $\omega \in \mathbb{R}$ such that, for any $t \in [a, b]$ and some $\theta \in (\frac{\pi}{2}, \pi)$, $\rho(A(t)) \supset \omega + \Sigma_\theta$, where $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$, and the resolvent estimate in $\omega + \Sigma_\theta$ is uniform with respect to $t \in [a, b]$;
- (ii) there exist a positive constant C and $0 \leq \theta_i < \alpha_i \leq 2$ ($i = 1, 2$) such that

$$\begin{aligned} & \| (A(t) - (\omega + 1)I)R(\lambda + \omega + 1, A(t))(R(\omega + 1, A(s)) - R(\omega + 1, A(t))) \|_{\mathcal{L}(C_b(\Omega))} \\ & \leq C \left(|\lambda|^{\theta_1 - 1} |t - s|^{\alpha_1} + |\lambda|^{\theta_2 - 1} |t - s|^{\alpha_2} \right) \end{aligned} \quad (\text{A.2})$$

for any $t, s \in [a, b]$, $\lambda \in \Sigma_\theta$.

Property (i) follows from the estimate proved by H.B. Stewart ([15]) in the autonomous case, noting that the constants appearing in the proof depend only on the ellipticity constant, the modulus of continuity and the L^∞ -norm of the coefficients. Such estimate shows that

$$\begin{aligned} & |\lambda| \|u\|_\infty + |\lambda|^{\frac{1}{2}} \|\nabla u\|_\infty + |\lambda|^{\frac{d}{2p}} \sup_{x_0 \in \overline{\Omega}} \|D^2 u\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda|}}(x_0))} \\ & \leq M \left(|\lambda|^{\frac{d}{2p}} \inf_{t \in [a, b]} \sup_{x_0 \in \overline{\Omega}} \|\lambda u - \mathcal{A}(t)u\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda|}}(x_0))} + |\lambda|^{\frac{1}{2}} \|g\|_\infty \right. \\ & \quad \left. + |\lambda|^{\frac{d}{2p}} \sup_{x_0 \in \overline{\Omega}} \|\nabla g\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda|}}(x_0))} \right), \end{aligned} \quad (\text{A.3})$$

for any λ , with $\text{Re} \lambda \geq \omega$, some $\omega > 0$, any function $u \in W_{\text{loc}}^{2,p}(\Omega) \cap C_b^1(\overline{\Omega})$, some $p > \max\{d/(2\alpha), d\}$. Here, g is any $W_{\text{loc}}^{1,p}(\Omega)$ -extension of $\mathcal{B}(t)u$, and M is a positive constant independent of λ , u and g . Since any operator $A(t)$ is sectorial, its resolvent set contains a right-halfline. Estimate (A.3) shows that $R(\cdot, A(t))$ is bounded in $\rho(A(t)) \cap \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq \omega\}$ and this implies that $\rho(A(t)) \supset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq \omega\}$ and $\|R(\lambda, A(t))\|_{\mathcal{L}(C_b(\overline{\Omega}))} \leq M|\lambda|^{-1}$ for any $\lambda \in \mathbb{C}$ with $\text{Re} \lambda \geq \omega$. Indeed, the norm of $R(\lambda, A(t))$ blows up as λ approaches the boundary of $\rho(A(t))$. (see e.g. [12, Prop. A.0.3]). Moreover, a simple argument based on von Neumann series and the previous estimate (see e.g., [12, Prop. 3.1.11]) shows that $\rho(A(t))$ contains the sector $\omega + \Sigma_\theta$ for $\theta = \pi - \arctan(2M) \in (\frac{\pi}{2}, \pi)$ and $\|R(\lambda, A(t))\|_{\mathcal{L}(C_b(\overline{\Omega}))} \leq 2M|\lambda - \omega|^{-1}$ for any $\lambda \in \omega + \Sigma_\theta$.

Property (ii) can be proved arguing as in [1, Thm. 6.3]. For the reader's convenience we enter into details and we prove it with $\alpha_1 = \alpha$, $\alpha_2 = \sigma$, $\theta_1 = d/(2p)$ and $\theta_2 = 1/2$ (note that our assumptions on σ and p guarantee that the conditions $\theta_1 < \alpha_1$ and $\theta_2 < \alpha_2$ are satisfied). Fix $f \in C_b(\overline{\Omega})$, $\lambda \in \mathbb{C}$ with positive real part, and let $v = R(\mu, A(s))f$ and $u = R(\lambda + \mu, A(t))(\lambda + \mu - A(s))R(\mu, A(s))f$, where $\mu = \omega + 1$. Clearly, $u - v = (A(t) - \mu I)R(\lambda + \mu, A(t))(R(\mu, A(s)) - R(\mu, A(t)))f$. So, if we set $w_{\lambda, \mu} = u - v$, estimate (A.2) becomes

$$\|w_{\lambda, \mu}\|_\infty \leq C \left(|\lambda|^{\frac{d}{2p} - 1} |t - s|^\alpha + |\lambda|^{-\frac{1}{2}} |t - s|^\sigma \right) \|f\|_\infty, \quad (\text{A.4})$$

for some constant C , independent of $f, \lambda, \alpha, \sigma, t, s$.

Applying estimate (A.3) to the function $w_{\lambda,\mu} \in C_b^1(\overline{\Omega}) \cap \bigcap_{p < +\infty} W_{\text{loc}}^{2,p}(\Omega)$ which satisfies the elliptic problem

$$\begin{cases} (\lambda + \mu)w_{\lambda,\mu} - \mathcal{A}(t)w_{\lambda,\mu} = (\mathcal{A}(t) - \mathcal{A}(s))v, & \text{in } \Omega, \\ \mathcal{B}(t)w_{\lambda,\mu} = [\mathcal{B}(s) - \mathcal{B}(t)]v, & \text{on } \partial\Omega, \end{cases}$$

we get

$$\begin{aligned} \|w_{\lambda,\mu}\|_\infty &\leq M_1 \left(|\lambda + \mu|^{\frac{d}{2p}-1} \sup_{x_0 \in \overline{\Omega}} \|(\mathcal{A}(t) - \mathcal{A}(s))v\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda+\mu|}}(x_0))} \right. \\ &\quad + |\lambda + \mu|^{\frac{d}{2p}-1} \sup_{x_0 \in \overline{\Omega}} \|\nabla_x(\mathcal{B}(s) - \nabla_x \mathcal{B}(t))v\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda+\mu|}}(x_0))} \\ &\quad \left. + |\lambda + \mu|^{-\frac{1}{2}} \|(\mathcal{B}(s) - \mathcal{B}(t))v\|_\infty \right), \end{aligned} \quad (\text{A.5})$$

for some positive constant M_1 , independent of f, λ, μ, t, s . From now on, we denote by L_j positive constants independent of $f, \lambda, \mu, \alpha, \sigma, t, s, x_0$. The smoothness of the coefficients imply that

$$\begin{aligned} &\|(\mathcal{A}(t) - \mathcal{A}(s))v\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda+\mu|}}(x_0))} \\ &\leq L_1 |t - s|^\alpha \left(\|D^2 v\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda+\mu|}}(x_0))} + |\lambda + \mu|^{-\frac{d}{2p}} \|v\|_{C_b^1(\overline{\Omega})} \right), \end{aligned} \quad (\text{A.6})$$

$$\|(\mathcal{B}(s) - \mathcal{B}(t))v\|_\infty \leq L_2 |t - s|^\sigma \|v\|_{C_b^1(\overline{\Omega})} \quad (\text{A.7})$$

$$\begin{aligned} &\|\nabla_x(\mathcal{B}(s)v - \mathcal{B}(t))v\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda+\mu|}}(x_0))} \\ &\leq L_3 |t - s|^\sigma \left(\|D^2 v\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda+\mu|}}(x_0))} + |\lambda + \mu|^{-\frac{d}{2p}} \|v\|_{C_b^1(\overline{\Omega})} \right). \end{aligned} \quad (\text{A.8})$$

Moreover, estimate (A.3) applied to $v = R(\mu, A(s))f$ shows that

$$\|v\|_{C_b^1(\overline{\Omega})} + \sup_{x_0 \in \overline{\Omega}} \|D^2 v\|_{L^p(\Omega \cap B_{1/\sqrt{|\lambda+\mu|}}(x_0))} \leq L_4 \|f\|_\infty, \quad (\text{A.9})$$

where we take into account that the ball $B_{1/\sqrt{|\lambda+\mu|}}(x_0)$ is contained into the ball $B_{1/\sqrt{|\mu|}}(x_0)$ since $\text{Re} \lambda > 0$ and $\mu > 0$. Replacing (A.6)-(A.9) into (A.5), taking into account that $|\lambda + \mu| \geq \mu > 1$ (which implies that $|\lambda + \mu|^{-1} \leq |\lambda + \mu|^{\frac{d}{2p}-1} \leq |\lambda + \mu|^{-\frac{1}{2}}$), that $|\lambda + \mu| \geq |\lambda|$ (since $\text{Re} \lambda > 0$) and our choice of p , we deduce estimate (A.4) in the halfplane $\{\lambda \in \mathbb{C} : \text{Re} \lambda \geq 0\}$.

To extend (A.4) to Σ_θ , we use again the proof of [12, Prop. 3.1.11] which shows that $R(\lambda + \mu, A(t)) = \sum_{n=0}^{+\infty} (-\text{Re} \lambda)^n R(\mu + i\text{Im} \lambda, A(t))^{n+1}$ for any $\lambda \in \Sigma_\theta$ with negative real part. Therefore,

$$\begin{aligned} \|w_{\lambda,\mu}\|_\infty &= \left\| \sum_{n=0}^{+\infty} (-\text{Re} \lambda)^n R(\mu + i\text{Im} \lambda, A(t))^n w_{\mu+i\text{Im} \lambda} \right\|_\infty \\ &\leq L_5 \left(|\text{Im} \lambda|^{\frac{d}{2p}-1} |t - s|^\alpha + |\text{Im} \lambda|^{-\frac{1}{2}} |t - s|^\sigma \right) \\ &\quad \times \sum_{n=0}^{+\infty} |\text{Re} \lambda|^n \|R(\mu + i\text{Im} \lambda, A(t))\|_{\mathcal{L}(C_b(\overline{\Omega}))}^n. \end{aligned}$$

To estimate the series, we recall that the choice of θ implies that $|\text{Re} \lambda| \leq (2M)^{-1} |\text{Im} \lambda|$ for any $\lambda \in \Sigma_\theta$. This and the resolvent estimate proved above show that $\sum_{n=0}^{+\infty} |\text{Re} \lambda|^n \|R(\mu + i\text{Im} \lambda, A(t))\|_{\mathcal{L}(C_b(\overline{\Omega}))}^n \leq 2$ for any $\lambda \in \Sigma_\theta$, and estimate (A.4) follows in the whole of Σ_θ . \square

Lemma A.2. Assume that Hypothesis 5.1(i) holds and that $\beta \in C_{\text{loc}}^{2+\alpha}(\overline{\Omega}, \mathbb{R}^d)$ is bounded together with all its derivatives on $\partial\Omega$. Then, there exists $r_0 > 0$ such that, for any $x_0 \in \partial\Omega$, there exists $\phi_{x_0} \in C^2(B_{r_0}(x_0), \mathbb{R}^d)$ such that

$$J\phi_{x_0}(x)\beta(x) = \rho_{x_0}(x)e_d, \quad x \in B_{r_0}(x_0) \cap \partial\Omega, \quad x_0 \in \partial\Omega, \quad (\text{A.10})$$

where $e_d = (0, \dots, 0, 1)^T$, for some continuous function ρ_{x_0} , which nowhere vanishes on $B_{r_0}(x_0) \cap \partial\Omega$. Moreover, $\phi_{x_0}(B_{r_0}(x_0) \cap \Omega)$ is a bounded domain contained in \mathbb{R}_+^d , $\phi_{x_0}(x) \in \overline{\mathbb{R}_+^d}$ if and only if $x \in B_{r_0}(x_0) \cap \overline{\Omega}$, $\phi_{x_0}(B_{r_0}(x_0) \cap \partial\Omega) \subset \overline{B_1^+} \cap \partial\mathbb{R}_+^d$ and there exists a positive constant Φ such that

$$\sup_{x_0 \in \partial\Omega} \left(\|\phi_{x_0}\|_{C^{2+\alpha}(B_{r_0}(x_0))} + \|\phi_{x_0}^{-1}\|_{C^{2+\alpha}(\phi_{x_0}(B_{r_0}(x_0)))} \right) \leq \Phi. \quad (\text{A.11})$$

Proof. Let $(B_R(x_h), \psi_h)_{h \in \mathbb{N}}$ be the covering of $\partial\Omega$ in Remark 2.2, with $x_h \in \partial\Omega$ and $R > 0$. For any $x = (x', x_d) \in \overline{B_1} = \psi_h(\overline{B_R(x_h)})$, we introduce the vector $\Upsilon_h(x) = J\psi_h(\psi_h^{-1}(x))\beta(\psi_h^{-1}(x', 0))$.

From now on, we fix an arbitrary $h \in \mathbb{N}$. We claim that the last component Υ_h^d of Υ_h nowhere vanishes on $\overline{B_1^+}$ or, equivalently, on $\overline{B_1^+} \cap \partial\mathbb{R}_+^d$. As it is easily seen,

$$\Upsilon_h^d(x) = \langle \nabla_x \psi_h^d(\psi_h^{-1}(x)), \beta(\psi_h^{-1}(x)) \rangle, \quad x \in \overline{B_1^+} \cap \partial\mathbb{R}_+^d. \quad (\text{A.12})$$

By Remark 2.2(c), $\nabla \psi_h^d(\psi_h^{-1}(x)) = -|\nabla_x \psi_h^d(\psi_h^{-1}(x))|\nu(\psi_h^{-1}(x))$ for any $x \in \overline{B_1^+} \cap \partial\mathbb{R}_+^d$. Formula (A.12) now shows that

$$|\Upsilon_h^d(x)| = |\nabla_x \psi_h^d(\psi_h^{-1}(x))| \langle \beta(\psi_h^{-1}(x)), \nu(\psi_h^{-1}(x)) \rangle, \quad x \in \overline{B_1^+} \cap \partial\mathbb{R}_+^d.$$

Recalling that β satisfies Hypothesis 2.3(iii) and the gradient of ψ_h nowhere vanishes in $B_R(x_h)$, we conclude that Υ_h^d nowhere vanishes on $\overline{B_1^+} \cap \partial\mathbb{R}_+^d$.

We can thus define the function $\tilde{\phi}_h : \overline{B_R(x_h)} \rightarrow \mathbb{R}^d$ by setting

$$\tilde{\phi}_h(x) = \left(\psi_h^1(x) - \frac{\Upsilon_h^1(\psi_h(x))}{\Upsilon_h^d(\psi_h(x))} \psi_h^d(x), \dots, \psi_h^{d-1}(x) - \frac{\Upsilon_h^{d-1}(\psi_h(x))}{\Upsilon_h^d(\psi_h(x))} \psi_h^d(x), \psi_h^d(x) \right),$$

for any $x \in \overline{B_R(x_h)}$. It is easy to notice that $\tilde{\phi}_h(B_R(x_h) \cap \partial\Omega) = \overline{B_1^+} \cap \partial\mathbb{R}_+^d$, that $\tilde{\phi}_h(B_R(x_h) \cap \Omega)$ is a bounded subset of \mathbb{R}_+^d and $\tilde{\phi}_h(x) \in \overline{\mathbb{R}_+^d}$ if and only if $x \in \overline{B_R(x_h)} \cap \overline{\Omega}$. Indeed ψ_h and $\tilde{\phi}_h$ agree on $B_R(x_h) \cap \partial\Omega$, $\tilde{\phi}_h^d \equiv \psi_h^d$ in $B_R(x_h) \cap \overline{\Omega}$ and $\psi_h^d(x) > 0$ (resp. $\psi_h^d(x) = 0$) if and only if $x \in B_R(x_h) \cap \Omega$ (resp. $x \in B_R(x_h) \cap \partial\Omega$). Moreover, $\tilde{\phi}_h \in C^{2+\alpha}(B_R(x_h) \cap \Omega)$ and $\sup_{h \in \mathbb{N}} \|\tilde{\phi}_h\|_{C^{2+\alpha}(B_R(x_h) \cap \Omega)} < +\infty$.

Let us now prove that

$$J\tilde{\phi}_h(x)\beta(x) = -\langle \beta(x), \nu(x) \rangle |\nabla \psi_h^d(x)| e_d, \quad x \in \partial\Omega \cap B_R(x_h). \quad (\text{A.13})$$

For this purpose, let us fix $x \in \partial\Omega \cap B_R(x_h)$. Since $\langle \beta(x), \nabla \psi_h^k(x) \rangle = \Upsilon_h^k(\psi_h(x))$ for any $k = 1, \dots, d$, it holds that

$$\begin{aligned} (J\tilde{\phi}_h(x)\beta(x))_k &= \langle \beta(x), \nabla \psi_h^k(x) \rangle - \frac{\Upsilon_h^k(\psi_h(x))}{\Upsilon_h^d(\psi_h(x))} \langle \beta(x), \nabla \psi_h^d(x) \rangle = 0, \quad k \leq d-1, \\ (J\tilde{\phi}_h(x)\beta(x))_d &= \langle \beta(x), \nabla \psi_h^d(x) \rangle = -\langle \beta(x), \nu(x) \rangle |\nabla \psi_h^d(x)|. \end{aligned}$$

Let us now recall that $\bigcup_{h \in \mathbb{N}} B_{R/2}(x_h) \supset \Omega_\varepsilon$ for some ε . We now want to prove that there exists $r_0 > 0$ such that, for any $h \in \mathbb{N}$ and any $x_0 \in \partial\Omega \cap B_{R/2}(x_h)$, the

function $\tilde{\phi}_h$ is invertible in $B_{r_0}(x_0)$. For this purpose, we observe that

$$J\tilde{\phi}_h(x) = J\psi_h(x) - \begin{pmatrix} \frac{\Upsilon_h^1(\psi_h(x))}{\Upsilon_h^d(\psi_h(x))} D_1 \psi_h^d(x) & \dots & \frac{\Upsilon_h^1(\psi_h(x))}{\Upsilon_h^d(\psi_h(x))} D_d \psi_h^d(x) \\ \vdots & \ddots & \vdots \\ \frac{\Upsilon_h^{d-1}(\psi_h(x))}{\Upsilon_h^d(\psi_h(x))} D_1 \psi_h^d(x) & \dots & \frac{\Upsilon_h^{d-1}(\psi_h(x))}{\Upsilon_h^d(\psi_h(x))} D_d \psi_h^d(x) \\ 0 & \dots & 0 \end{pmatrix},$$

for any $x \in B_R(x_h) \cap \partial\Omega$. Hence, $\det(J\tilde{\phi}_h(x)) = \det(J\psi_h(x)) \neq 0$ for such x 's. Now, Remark 2.2 shows that there exists a positive constant C , independent of h , such that $\det(J\tilde{\phi}_h(x)) \geq C$ for any $x \in B_R(x_h) \cap \partial\Omega$. The inverse mapping theorem and, again the equiboundedness of the norms the functions ψ_h and ψ_h^{-1} show that we can determine $r_0 \in (0, R/2)$, independent of h , such that $\tilde{\phi}_h$ is invertible in $B_{r_0}(x_0)$ for any $x_0 \in \partial\Omega \cap B_{R/2}(x_h)$ and its inverse map belongs to $C^{2+\alpha}(\phi_h(B_{r_0}(x_0) \cap \Omega))$ with $C^{2+\alpha}$ -norm bounded uniformly with respect to h and x_0 . We set $\phi_{x_0, h} = (\tilde{\phi}_h)|_{B_{r_0}(x_0) \cap \Omega}$.

For any $x_0 \in \partial\Omega$, we denote by $h(x_0)$ the smallest integer such that $x_0 \in B_{R/2}(x_h)$, and we define $\phi_{x_0} = \phi_{x_0, h(x_0)}$. From the previous results, we know that the family $\{\phi_{x_0} : x_0 \in \partial\Omega\}$ satisfies (A.11). Moreover, formula (A.13) yields (A.10) with $\rho_{x_0}(x) = -|\nabla \psi_{h(x_0)}^d(x)| \langle \beta(x), \nu(x) \rangle$. \square

Lemma A.3. *For any $0 < r_1 < r_2 < r_0$ and any $x_0 \in \partial\Omega$, there exist a function $\vartheta \in C_c^\infty(\mathbb{R}_+^d)$ and a positive constant C such that $\chi_{\phi_{x_0}(B_{r_1}(x_0) \cap \Omega)} \leq \vartheta \leq \chi_{\phi_{x_0}(B_{r_2}(x_0) \cap \Omega)}$, $D_d \vartheta \equiv 0$ on $\partial\mathbb{R}_+^d$ and $\|D^k \vartheta\|_\infty \leq K(r_2 - r_1)^{-k}$ for $k = 0, 1, 2, 3$ and some positive constant K , where ϕ_{x_0} and r_0 are as in Lemma A.2.*

Proof. We fix r_1, r_2 as in the statement and let $\varepsilon = \frac{r_2 - r_1}{6\Phi}$, where Φ is defined in (A.11), and let ϑ_0 be the convolution of a standard mollifier, supported in the ball B_ε , and the characteristic function of the set $\{x \in \mathbb{R}^d : d(x, E) \leq \varepsilon\}$, where $E = \{x \in \mathbb{R}^d : (x', |x_d|) \in \phi_{x_0}(B_{r_1}(x_0) \cap \Omega)\}$. The function ϑ_0 is smooth and even with respect to the last coordinate, hence $D_d \vartheta_0 \equiv 0$ on $\partial\mathbb{R}_+^d$. We set $\vartheta = (\vartheta_0)|_{\mathbb{R}_+^d}$. Clearly, $\vartheta \equiv 1$ in $\phi_{x_0}(B_{r_1}(x_0) \cap \Omega)$ and its support is contained in the set $F = \{x \in \mathbb{R}_+^d : d(x, \phi_{x_0}(B_{r_1}(x_0) \cap \Omega)) \leq 2\varepsilon\}$. We claim that $F \subset \phi_{x_0}(\overline{B_{\frac{r_1+r_2}{2}}(x_0)} \cap \overline{\Omega})$. Note that it is enough to prove that $F \subset \phi_{x_0}(\overline{B_{\frac{r_1+r_2}{2}}(x_0)})$. Indeed, from the definition of the function ϕ_{x_0} it follows that $\phi_{x_0}(\overline{B_{\frac{r_1+r_2}{2}}(x_0)}) \cap \mathbb{R}_+^d = \phi_{x_0}(\overline{B_{\frac{r_1+r_2}{2}}(x_0)} \cap \overline{\Omega})$. Since $|x - x'| = |\phi_{x_0}^{-1}(\phi_{x_0}(x)) - \phi_{x_0}^{-1}(\phi_{x_0}(x'))| \leq \Phi |\phi_{x_0}(x) - \phi_{x_0}(x')|$ for any $x \in \partial B_{r_1}(x_0)$ and $x' \in \partial B_{\frac{r_1+r_2}{2}}(x_0)$, we immediately deduce that

$$d(\phi_{x_0}(\partial B_{r_1}(x_0)), \phi_{x_0}(\partial B_{\frac{r_1+r_2}{2}}(x_0))) \geq \frac{r_2 - r_1}{2\Phi} = 3\varepsilon.$$

Noting that

$$\begin{aligned} d(\phi_{x_0}(B_{r_1}(x_0)), \mathbb{R}^d \setminus \phi(B_{\frac{r_1+r_2}{2}}(x_0))) &= d(\partial\phi_{x_0}(B_{r_1}(x_0)), \partial\phi_{x_0}(B_{\frac{r_1+r_2}{2}}(x_0))) \\ &= d(\phi_{x_0}(\partial B_{r_1}(x_0)), \phi_{x_0}(\partial B_{\frac{r_1+r_2}{2}}(x_0))), \end{aligned}$$

we conclude that $d(x, \phi_{x_0}(B_{r_1}(x_0))) \geq 3\varepsilon$ for any $x \in \mathbb{R}^d \setminus \phi(B_{\frac{r_1+r_2}{2}}(x_0))$ and this shows that $F \subset \phi_{x_0}(\overline{B_{\frac{r_1+r_2}{2}}(x_0)})$ as claimed. Indeed, if $x \in F$, it holds that $d(x, \phi_{x_0}(B_{r_1}(x_0))) \leq d(x, \phi_{x_0}(B_{r_1}(x_0) \cap \Omega)) \leq 2\varepsilon$. It thus follows that $\text{supp } \vartheta \subset \phi_{x_0}(B_{r_2}(x_0) \cap \overline{\Omega})$.

Finally, we observe that $\|D^k \vartheta\|_\infty \leq M\varepsilon^{-k}$ for $k = 0, 1, 2, 3$ and some positive constant M . Our choice of ε leads to the estimates in the statement. \square

Lemma A.4. *Let $f : [a, b] \times \Omega \rightarrow \mathbb{R}$ be a bounded and continuous function and let $g : \Omega \rightarrow \mathbb{R}$ be the function defined by $g(x) = \int_a^b f(t, x)dt$ for any $x \in \Omega$. Then, $g \in C_b(\Omega)$. Moreover, for any bounded linear operator $T : C_b(\Omega) \rightarrow C_b(\Omega)$, which transforms bounded sequence of continuous functions, converging locally uniformly in Ω , into sequences with the same properties, it holds that $(Tg)(x) = \int_a^b (Tf(t, \cdot))(x)dt$ for any $x \in \Omega$.*

Proof. Showing that $g \in C_b(\Omega)$ is an easy task left to the reader.

To prove the last part of the proof, for any $n \in \mathbb{N}$, let $g_n = \sum_{k=0}^{n-1} f(t_k, \cdot)(t_{k+1} - t_k)$, where $t_k = a + k(b - a)/n$ for $k \in \{0, \dots, n\}$. Clearly, g_n converges to g locally uniformly in Ω . Further, $\|g_n\|_\infty \leq \|f\|_\infty(b - a)$ for any $n \in \mathbb{N}$. Hence, Tg_n converges to Tg locally uniformly in Ω as $n \rightarrow +\infty$. As is immediately seen, $(Tg_n)(x) = \sum_{k=0}^{n-1} (Tf(t_k, \cdot))(x)(t_{k+1} - t_k)$ for any $x \in \Omega$ and $n \in \mathbb{N}$. Since the function Tf is continuous in $[a, b] \times \Omega$, the same arguments as above show that $(Tg_n)(x)$ converges to $\int_a^b (Tf(s, \cdot))(x)ds$ as $n \rightarrow +\infty$, and we are done. \square

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